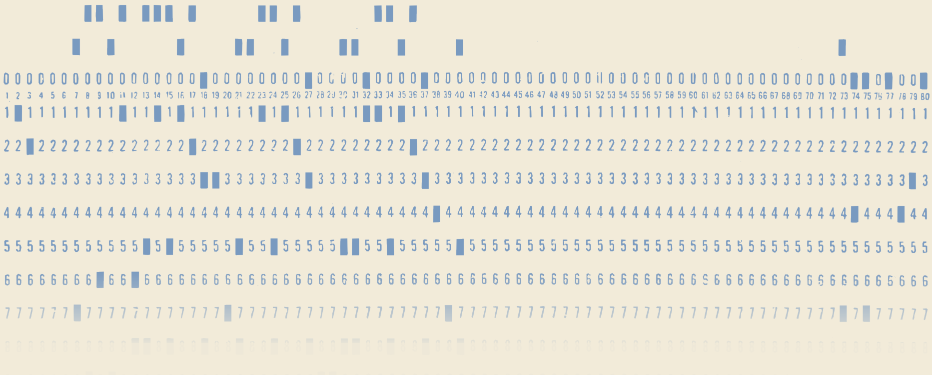

Lecture 18: Parallel Algorithms

COMP526: Efficient Algorithms

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University of Liverpool

Updated: December 3, 2024

Announcements

1. Quiz 07 on Error Correcting Codes
   - Complete by 11:59pm, Friday 06 November
2. Grading is slow (sorry)
3. Last lectures:
   - Parallel Algorithms (today)
   - Text indexing (Thursday, next Tuesday)
   - Final review (next Thursday)
4. Attendance Code:

Meeting Goals

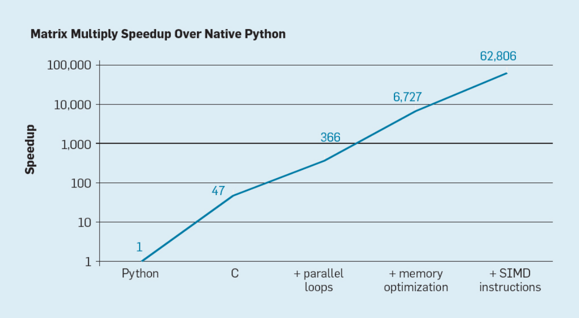
1. Discuss parallel algorithms!
2. Formalize cost measures for parallel algorithms
3. Argue Brent's theorem
4. Describe parallel searching algorithms
5. Describe parallel sorting algorithms
    - Sorting networks
    - Parallel MergeSort

Parallel Algorithms

From Last Time

Parallel Algorithms

- Modern computers can perform many operations simultaneously
- **SIMD**: single instruction, multiple data (e.g., GPU)
- **MIMD**: multiple instructions, multiple data (e.g., multicore CPU)
- To achieve maximal performance, parallelism of hardware must be exploited



PRAM Model

Parallel RAM

- Unbounded number of **processing elements** (PEs) think **cores**
- Access *shared memory*

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Bottom Line. No single model is well-suited for all applications

- we'll assume CREW
- reasoning about parallel programs can be **incredibly subtle!**

Measuring PRAM Efficiency

Main cost metrics

- **space:** the total amount of accessed memory
- **time:** the number of steps until all processes terminate
  - also known as **depth** or **span**
- **work:** total number of instructions executed by all processes

Measuring PRAM Efficiency

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- **work:** total number of instructions executed by all processes

Goal:

- minimal span (= time)
- work is (asymptotically) no worse than the best *sequential* algorithm
  - called **work-efficient** algorithms

Models vs Reality

Idealization. The PRAM model does not limit the number of possible PEs (processing elements)

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Questions.

- How relevant/applicable is the PRAM model if it assumes access to an *unlimited* number of PEs?
- Can **every** task be performed efficiently in PRAM?
  - are there problems that are *inherently sequential?*

Brent's Theorem

Theorem (Brent). If an algorithm has span T and work W for an arbitrary number of processors, then the algorithm can be run on a PRAM with p PEs in time $O(T + W/p)$ using work W.

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- Proof Idea: schedule parallel steps in a "round-robin" fashion on the p PEs.

Enough Generalities!

Parallel Algorithms

- Searching
- Sorting
  - Sorting Networks (SIMD)
    - sorting short lists
  - Parallel MergeSort
    - sorting long lists

Parallel Searching

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    - not obvious how to employ parallelism
  - LZW compression ("P-complete")
    - Input: string S phrase p
    - Output: does LZW add p to the dictionary?

Parallel String Matching

Recall the **string matching** problem:

- Text T, length n
- Pattern P, length m
- **Goal:** find all occurrences of P in T
  - return array M of length n where $M[i] = 1$ if P matches T at index i, and $M[i] = 0$ otherwise

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Question. Is this problem embarrassingly parallel?

- can check each index i independently!

Parallel String Matching: Brute Force

Idea. Use the *brute force* procedure to check each i in parallel.

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procedure ParallelBFMatch(T[0..n), P[0..m))
    for i = 0, 1, ..., n − 1 in parallel do
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            if T[i + j] ≠ P[j] then break
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PollEverywhere

What is the **span** of this computation?

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Question. Can we do better?

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- compute failure link array
- apply FLA to search for matches

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PollEverywhere

What is the running time of KMP to search for a pattern of length m in a text of length n?

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procedure $\textsc{ParallelKMP}(T[0..n), P[0..m))$
  for $b = 0, 1, \ldots, \lceil n/m \rceil$ **in parallel do**
    $T_b = T[mb, mb + 2m - 1)$
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- Span: $O(m)$
- Work: $O(n)$
  - this is work efficient!

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Parallel String Matching Discussion

Assessment

- very simple methods
- can be run in a *distributed* setting
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Questions

- What if we only want to find if there is a single occurrence of P in T?
- What if m large? State of the art:
    - $O(\log m)$ & work efficient for CREW-PRAM
    - CRCW-PRAM $O(1)$ matching part in $O(1)$ time, with $\Theta(\log\log m)$ preprocessing

Sorting Networks

Comparitors

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    for $i = 1, 2, \ldots, n-1$ **do**
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Abstract View. A **comparator** is is a PE that takes two values as inputs and returns the values in sorted order.

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Question. Which comparator operations of InsertionSort can be performed in parallel (while still ensuring correct output)?

Comparator Networks

Visual Representation.

- Inputs/indices are represented by **wires** (horizontal lines)
- Comparators are vertical line segments between wires
  - interpretation: wire between wire i and j performs comp to indices i and j input
- Execution: Scan diagram from left to right and apply comparators in order they are encountered

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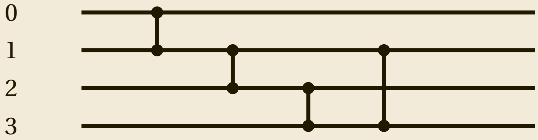
Example. Consider the following comparator network on 4 wires. What is the output on input $[4, 3, 2, 1]$?



Sorting Algorithms to Networks

Consider InsertionSort on inputs of size 5. What are the (possible) comparator operations performed by the algorithm?

- Which comparator operations could be performed *in parallel*?

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Definitions.

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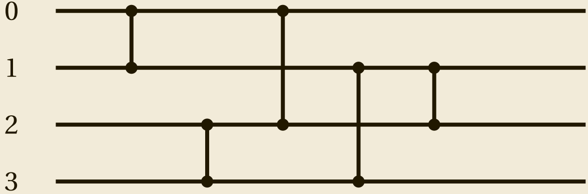
Question. What is the smallest/shallowest sorting network for a given input size?

- Optimal *size* sorting networks are only known for up to 12 inputs
- Optimal *depth* is only known up to 18 inputs

Some Optimal Sorting Networks

Example. $n = 4$ wires. What is the depth?



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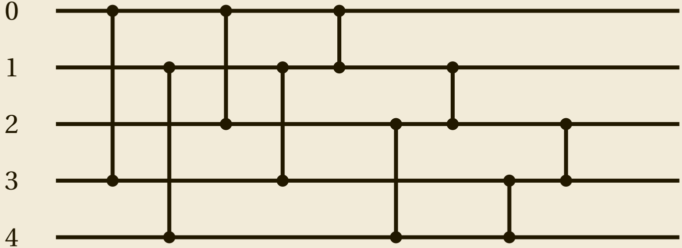

PollEverywhere

What is the **depth** of this sorting network?

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Sorting Network Discussion

Applications

- Hardware-level implementations
    - comparators implemented with simple circuits
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General Construction. Bitonic Merge Sort

- Mimic recursive structure of MERGESORT
- Size $O(n \log^2 n)$
- Depth $O(\log^2 n)$
- Not work-efficient, but still practical

Parallel MergeSort

Parallel Divide & Conquer?

Observation. The Divide & Conquer strategy can lend itself well to parallelism:

1. Divide problem into sub-tasks
2. Solve the subtasks

3. Merge solutions of the subtasks

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2. Solve the subtasks (**independently**)
   - Parallelize these!
3. Merge solutions of the subtasks (**...?**)
   - How to parallelize this?

Parallel MergeSort?

Revisited: MergeSort

```
procedure MergeSort(A, i, k)
    if i < k then
        j ← ⌊(i + k)/2⌋
        MergeSort(A, i, j)
        MergeSort(A, j + 1, k)
        B ← Copy(A, i, j)
        C ← Copy(A, j + 1, k)
        Merge(B, C, A, i)
    end if
end procedure
```

Parallel MergeSort?

PollEverywhere

What is the span of MergeSort with parallel recursive calls and sequential merges?

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procedure MergeSort(A, i, k)
  if $i < k$ **then**
    $j \leftarrow \lfloor (i+k)/2 \rfloor$
    MergeSort(A, i, j)
    MergeSort($A, j+1, k$)
    $B \leftarrow$ Copy(A, i, j)
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Idea.

- In x's own sub-array, just use x's index!

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- How do we find this?
- index = # elements $\leq x$
  - # in x's sub-array
  - # in other sub-array
- How to compute these?

Idea.

- In x's own sub-array, just use x's index!
- For the other sub-array, use binary search!

Parallelizing Merges

Question. How can we *parallelize merges?*

- For each x, find the final index of x
- How do we find this?
- index = # elements $\leq x$
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  - # in other sub-array
- How to compute these?

Idea.

- In x's own sub-array, just use x's index!
- For the other sub-array, use binary search!
- **Parallelize:** do each x in parallel!

Parallel MergeSort in Code

```
procedure ParallelMerge(A[l..m), A[m..r), B)
    for i = l, ..., m − 1 in parallel do
        k ← (i − l) + BinarySearch(A[m..r), A[i])
        B[k] ← A[i]
    end for
    for j = m, m + 1, ..., r − 1 in parallel do
        k ← BinarySearch(A[l..m), A[j])
        B[k] ← A[j]
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Questions.

- What is the **span** of ParallelMerge?

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Questions.

- What is the **span** of ParallelMerge?
    - $\Theta(\log n)$
- What is the **work** of ParallelMerge?

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Parallel MergeSort Analysis

Overall Procedure

1. Split (sub)array in half
2. Parallel recursive MergeSorts
3. ParallelMerge sorted halves

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Improvements. Merge can be improved to $\Theta(n)$ work! (but it's complicated)

Concluding Thoughts

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- Computer hardware is naturally parallel
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Parallelism is Subtle

- Reasoning about parallel programs is hard
- Writing correct parallel programs is hard
- Idealized models abstract away many challenges
    - no perfect synchrony?
    - tolerate faults?

Next Time

- Text Indexing

Scratch Notes