Lecture 17: Error Correction; Parallel Algorithms

COMP526: Efficient Algorithms

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Announcements

1. Programming Assignment 2 posted
   - Due 29 November
2. No Quiz This Week!
3. Attendance Code:

Meeting Goals

1. Finish discussion error correcting codes
   - Parity checking
   - Hamming Codes
2. Introduce parallel algorithms

Error Correcting Codes

From Last Time

Communcation Model.

- Goal: send a text $S \in \{0,1\}^*$ (bitstream) across a communication channel
- Any bit transmitted through the channel might **flip**
  - $0 \mapsto 1$ or $1 \mapsto 0$
  - *no* erasures or insertions
- To cope with errors:
  - compute and send an encoded bitstream $C(S)$
  - receiver decodes C to get S

Block Codes. Assumptions

- Messages consists of fixed sized blocks
  - k = **message length**
  - $m \in \{0,1\}^k$
- Encode each message separate as $C(m) \in \{0,1\}^n$
  - $C(m)$ is **codeword** for m
- n is the **block length**

Requirements for Detecting and Correcting

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Correcting Requirement. Suppose C can correct errors of flipping up to b bits. Then C has distance $d \geq 2b+1$

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Proof sketch.

- Consider the deleting the first $d - 1$ bits of each codeword.
- Remaining codewords are still pair-wise distinct
- There are only $2^{n-(d-1)}$ possible shorter bitstrings

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Hamming bound. $2^k \leq 2^n / \sum_{f=0}^{\lfloor (d-1)/2 \rfloor} \binom{n}{f}$.

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Proof sketch.

- Codewords must be at distance d away
- Thus balls centered at codewords of radius $\lfloor (d-1)/2 \rfloor$ must be disjoint
- Number of balls × *volume* of each ball must be at most 2^n

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Question. These are *im*possibility results. What is possible?

Error Detection & Correction

Error Detection: Parity Bits

Question. How can we **detect** a single error?

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Idea. Form C by adding an extra bit to message m that encodes the parity of m

- the extra bit is called a **parity bit**
- which strings are valid codewords?

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- the extra bit is called a **parity bit**
- which strings are valid codewords?
  - the parity of valid codewords is always 1!

Parity Bit Example

Small Example. Consider $k = 2$, so that $n = 3$ with parity bits.

- Messages $\{00, 01, 10, 11\}$

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PollEverywhere Question

Consider the code C with $k = 2$ bit messages and one parity bit. What is the distance d of C?

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- $\mathscr{C} = \{000, 011, 101, 110\}$
- What is the distance of C?
- How do we detect errors?

Error Correction through Duplication

Suppose we want to **correct** a single error. How is this even possible?

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Simple Solution. Duplicate each bit 3 times and send the duplicates!

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- How do we decode a message?

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- View on Hamming cube!

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Inefficiency. To correct a single error, we must **triple** the length of the message?!

A Puzzle. How can we correct a single error more efficiently?

- Don't need to duplicate every bit!
- Idea: use parity checks on *parts* of the string to identify the index where error occurred!

Hamming Codes

How to Locate Errors?

Idea. Use several parity bits!

- Each parity bit detects an error on a part of the input
- Choose parts so that parity checks uniquely specify location of error
- Error may be in one of the parity bits itself!

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Binary Trick. Blocks of size $n = 7$ bits: $B = B_7 B_6 B_5 B_4 B_3 B_2 B_1$

- Write indices in binary
  - `111, 110, 101, 100, 011, 010, 001`
- Have a parity check for each bit of the index where the error could have occurred
  - was the error at an index whose jth bit is 1?
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- **Question.** Where do we store parity/message bits?

(7, 4) Hamming Code

Parity Values. Store parity bits at indices $j = \texttt{100}_2, \texttt{010}_2, \texttt{001}_2$.

- Use other 4 bits for messages

index	111	110	101	100	011	010	001
bit	B_7	B_6	B_5	B_4	B_3	B_2	B_1

Question. Why use these three bits for parity checks?

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- They are independent of the other parity checks!
  - 111, 110, 101, 100, 011, 010, 001
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Encoding (7, 4) Hamming Code

Procedure. To encode $m = m_3 m_2 m_1 m_0$:

1. write the bits of m to indices $7, 6, 5, 3$ of the codeword

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   - $p_4 = m_3 \oplus m_2 \oplus m_1$
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index	111	110	101	100	011	010	001
bit	m_3	m_2	m_1	p_4	m_0	p_2	p_1

Example. Encode the message $m = 1011$

index	111	110	101	100	011	010	001
bit							

Hamming Code Distance

Recall. **Code distance** is the minimum Hamming distance between any two codewords.

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PollEverywhere Question

What is the code distance of the (7,4) Hamming code?

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- Suppose $A = A_7 A_6 A_5 A_4 A_3 A_2 A_1$ and $B = B_7 B_6 B_5 B_4 B_3 B_2 B_1$ are codewords
- A_4, A_2, A_1 determined from other values (similarly for B)
- A and B differ on at least one index $7 = 111_2, 6 = 110_2, 5 = 101_2, 3 = 011_2$
- If A and B differ on exactly one message bit, then two parity bits differ as well
- Check: if A and B differ on two message bits, then at least one parity bit differs as well!

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Note. Code distance 3 implies correcting 1 error *might* be possible…

Decoding (7, 4) Hamming Code

Procedure. Given received message $B = B_7 B_6 B_5 B_4 B_3 B_2 B_1$:

1. Compute the parity bits
   - $p_4 = B_7 \oplus B_6 \oplus B_5 \oplus B_4$
   - $p_2 = B_7 \oplus B_6 \oplus B_3 \oplus B_2$
   - $p_1 = B_7 \oplus B_5 \oplus B_3 \oplus B_1$
2. Form index j with binary representation $p_4 p_2 p_1$
3. If $j \neq 0$, form B' by flipping B_j to $1 - B_j$
4. Decode the message $m = B'_7 B'_6 B'_5 B'_3$

Example. Decode the message $B = \texttt{1110101}$

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Example. Decode the message $B = \texttt{1110101}$

- $m = \texttt{1011}$

Note. If $j = 0$, then B is a valid codeword. If $j \neq 0$, then B' is a valid codeword at distance 1 from B.

Error Correction Prospectus

(7, 4) Hamming Codes are **perfect**:

- m, n, and d match the Hamming *lower bound* for block codes

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Generalizations.

- General Hamming codes:
  - Codeword length $n = 2^\ell - 1$ for any ℓ
  - ℓ parity bits
  - Message length $2^\ell - \ell - 1$ message length
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  - All are perfect!
- Other optimal values of m, n, d are generally not known
  - many efficient schemes use algebraic constructions
  - almost all randomly chosen codes are good(!)
  - ongoing research!

Parallel Algorithms

Improving Technology?

Laptop Power.

- My first laptop (ca. 2004)
  - Compaq Presario 2100
  - $900 new ($1,500 with inflation)
  - now < $15 used
- Recent laptop (ca. 2021)
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PollEverywhere Question

How much *faster* is a new mid/high range laptop computer today than a comparable model from 20 years ago?

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Processor Speed is Not Increasing

Year	Transistors	Clock speed	CPU model
1979	30 k	5 MHz	8088
1985	300 k	20 MHz	386
1989	1 M	20 MHz	486
1995	6 M	200 MHz	Pentium Pro
2000	40 M	2 000 MHz	Pentium 4
2005	100 M	3 000 MHz	2-core Pentium D
2008	700 M	3 000 MHz	8-core Nehalem
2014	6 B	2 000 MHz	18-core Haswell
2017	20 B	3 000 MHz	32-core AMD Epyc
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But the **number of transistors** is growing exponentially!

Speed vs Throughput

Measuring Performance

- **Processor speed** is the number processor clock cycles per second
- **Latency** of an operation is the time from when the operation starts to when it completes
  - speed determines latency of individual operations
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- **WRONG!!!**
  - how long does a degree take?
  - how does U of L have so many graduates?

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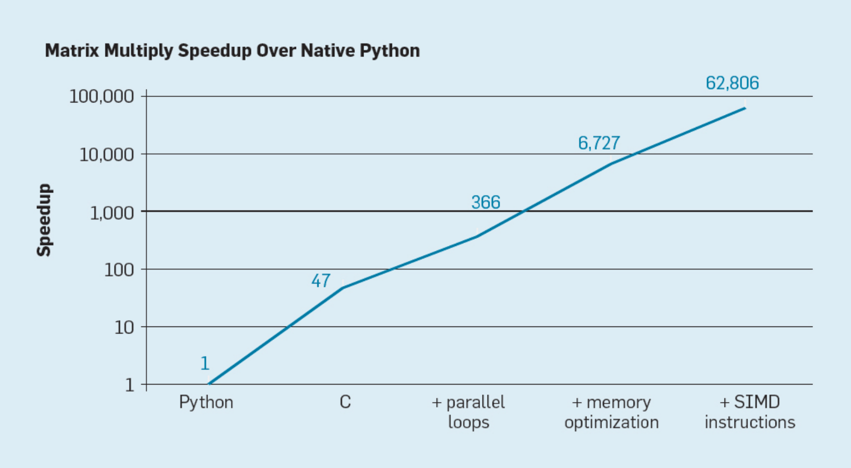
- Multiple Instruction Multiple Data (MIMD) parallelism:
  - multicore CPUs
- Distributed/networked computing
  - cluster computing, “cloud” computing, server farms

The Power of Parallelism



Modeling Parallel Computing

Restricted Model: SIMD instructions

- Program = sequence of instructions to be performed
- If *same* operation is performed on multiple data, operations can be performed simultaneously
- Example:

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- Program can spawn processes/processing elements (PEs) that run in parallel
  - each process is like its own program
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Warning. PRAM programs can be *incredibly subtle* to reason

Measuring PRAM Efficiency

Main cost metrics

- **space:** the total amount of accessed memory
- **time:** the number of steps until all processes terminate
  - also known as **depth** or **span**
- **work:** total number of instructions executed by all processes

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Goal:

- minimal span (= time)
- work is (asymptotically) no worse than the best *sequential* algorithm
  - called **work-efficient** algorithms

Models vs Reality

Idealization. The PRAM model does not limit the number of possible PEs (processing elements)

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- Idea: schedule parallel steps in a "round-robin" fashion on the p PEs.

Enough Generalities!

Parallel Algorithms

- Sorting
  - Sorting Networks (SIMD)
    - sorting short lists
  - Parallel MergeSort
    - sorting long lists
- Searching

Sorting Networks

Comparitors

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Example. INSERTIONSORT

```
procedure INSERTIONSORT(a, n)
    for i = 1, 2, ..., n − 1 do
        j ← i
        while j > 0 and a[j] < a[j − 1] do
            SWAP(a, j, j − 1)
            j ← j − 1
        end while
    end for
end procedure
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Abstract View. A **comparator** is is a PE that takes two values as inputs and returns the values in sorted order.

- $\text{comp}(x, y) = (\min\{x, y\}, \max\{x, y\})$
- all array modifications of InsertionSort can be performed by comparators

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Example. InsertionSort

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procedure InsertionSort(a, n)
    for i = 1, 2, ..., n − 1 do
        j ← i
        while j > 0 and a[j] < a[j − 1] do
            Swap(a, j, j − 1)
            j ← j − 1
        end while
    end for
end procedure
```

Abstract View. A **comparator** is is a PE that takes two values as inputs and returns the values in sorted order.

- $\text{comp}(x, y) = (\min\{x, y\}, \max\{x, y\})$
- all array modifications of InsertionSort can be performed by comparators

Question. Which comparator operations of InsertionSort can be performed in parallel (while still ensuring correct output)?

Comparator Networks

Visual Representation.

- Inputs/indices are represented by **wires** (horizontal lines)
- Comparators are vertical line segments between wires
  - interpretation: wire between wire i and j performs comp to indices i and j input
- Execution: Scan diagram from left to right and apply comparators in order they are encountered

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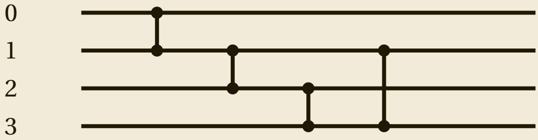
Example. Consider the following comparator network on 4 wires. What is the output on input $[4,3,2,1]$?



Sorting Algorithms to Networks

Consider InsertionSort on inputs of size 5. What are the (possible) comparator operations performed by the algorithm?

- Which comparator operations could be performed *in parallel*?

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Sorting Network Terminology

Definitions.

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Sorting networks and parallel algorithms.

- Each comparator is a process element
- The depth is the span (running time) of the network
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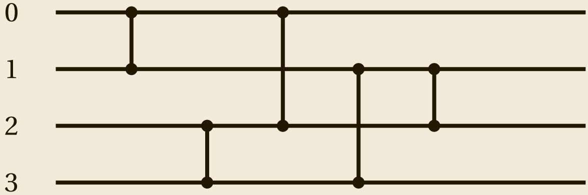
Question. What is the smallest/shallowest sorting network for a given input size?

- Optimal *size* sorting networks are only known for up to 12 inputs
- Optimal *depth* is only known up to 18 inputs

Some Optimal Sorting Networks

Example. $n = 4$ wires. What is the depth?



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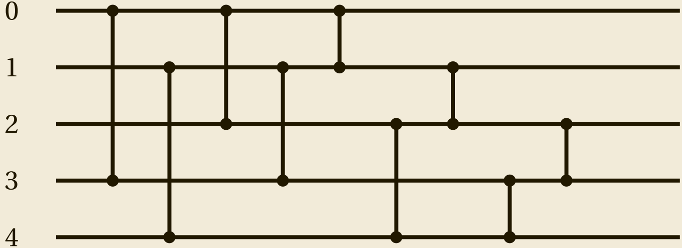

Next Time

- More parallel sorting!
- Parallel searching!

Scratch Notes