Lecture 14: Data Compression II

COMP526: Efficient Algorithms

Updated: November 19, 2024

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Announcements

1. Programming Assignment 2 posted
    - Due 29 November
2. Quiz 6 due Friday
    - Covers Lecture 13 material
    - 1 Question, Short Answer
    - Usual rules apply
3. Attendance Code:

Meeting Goals

1. Introduce Programming Assignment 2
2. Discuss limitations of general compression
3. Introduce compression techniques that exploit redundancy in texts
    3.1 Run length encoding
    3.2 Elias codes
    3.3 Lempel-Ziv-Welch (LZW) encoding

Programming Assignment 2

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  - learns correct answers immediately before exam
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- Before the exam:
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Goal: figure out a scheme to get the highest possible **guaranteed** score (without knowing how to answer any questions correctly yourself)

- For all possible (correct) exam solutions, maximize the *worst* score you receive

The Problem, Formalized

Three Pieces:

- $B = B[0..20)$ the correct solutions to the exam
  - expressed in binary 1 for true, 0 for false
  - known only to your hacker friend
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One Goal: Achieve the maximum *guaranteed* score.

- $20 - \max\left\{d_H(A,B) \,\middle|\, B \in \{0,1\}^{20}\right\}$
- $d_H(A,B)$ is **Hamming distance** = number of indices where solutions differ

Your Task

Main Task. Implement functions to compute & decode the message M

- complete `exam_cheat_code.py`
- `encode(solutions: list[int]) -> list[int]`
  - input: list of binary values, B, length 20
  - output: list of binary values, M, length 10
- `decode(message: list[int]) -> list[int]`
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Secondary Task. Explain how your scheme works!

- complete a single page PDF (typed) explaining your approach

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Optional Task. Prove an **upper bound** on the best achievable score for **any** cheating scheme.

Evaluation

Total marks: 100

- Main Task (code): 70 marks
  - higher guaranteed test score = more marks!
  - > 70 marks possible if guaranteed score is > 16
  - more marks for **simpler** solutions (tie breaking)
- Secondary Task (explanation): 30 marks
  - Concise and clear explanation of approach
  - Sensible/systematic approach
- Optional Task (upper bound proof): up to 20 marks **extra credit**

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Admininstration

- Full instructions on course website: https://willrosenbaum.com/teaching/2024f-comp-526/
- Submission through Canvas
- **Due 29 November** (next Friday)

Limits of Compression

From Last Time

Introduced lossless compression task

- Compression ratio
  - $\frac{|C|\cdot\log|\Sigma_C|}{\left|S\cdot\log|\Sigma_S|\right|} \overset{\Sigma_C=\{0,1\}}{=} \frac{|C|}{|S|\cdot\log|\Sigma_S|}$
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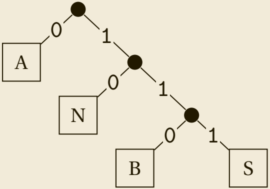

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$E(c)$	0	10	110	111

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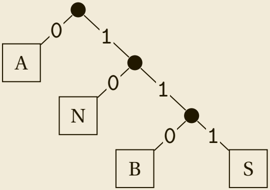

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- Prefix coding
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- Huffman codes
  - most efficient possible prefix code

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PollEverywhere Question

Suppose S is a text of length n over an alphabet Σ_S of size 8. What is the **smallest** possible compression ratio of any character encoding of S?

`pollev.com/comp526`

Character Encoding Limitations

An Issue with character encodings:

- Only single characters are encoded *in isolation*
- Cannot exploit *larger patterns* in text

Example. Huffman encoding doesn't distinguish between the following texts:

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- One idea: use a larger source alphabet—e.g., use pairs of characters

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Example.

```
s = ‘’
for i in range(1000000):
  s = s + ‘A’
print(s)
```

Quantifying General Compression

Definition. Suppose we fix a (programming) language L (e.g., Python). Given a source text S, the **Kolmogorov complexity** of S (relative to L), denoted $K(S)$ is the length of the shortest program whose output is S.

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- S may admit a huge amount of compression
    - $S = \underbrace{\texttt{AAA...A}}_{n\text{ times}}\underbrace{\texttt{BBB...B}}_{2n\text{ times}}$
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Question. How much compression can we achieve in this way?

- How close to $K(S)$ can we get?

Limits of General Compression

Fact 1. Suppose $\Sigma = \{0,1\}$ and fix any language L. Then for every positive integer n, there exists a source text $S \in \Sigma^n$ for which $K(S) \geq n$.

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Well we can't compress *everything*, but how well can we do?

- Can we generally find an optimal compression of a string in a given language?
- We did manage this for prefix codes! (Huffman codes)

Impossibility and Compression

Theorem

Suppose L is a "sufficiently rich" programming language (e.g. Python). Then there is no algorithm/program that for any string S:

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- **Claim.** $K(S_n) = O(\log n)$.
- This contradicts the assumption that P computed $K(S)$. $\square$

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Moral. There is no general method for determining how compressible a source text might be.

- Most texts are not very compressible.
    - Generalization of Fact 1.
- But many "interesting" source texts are compressible.
- Can still exploit features of common texts
    - most "interesting" source texts obey some patterns

Run Length Encoding

A Simple Idea

Example. How could we compress the following source text?

```
000000000000000000000000000000000000000
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000101100100000111111000000000011111000
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001100000000000000000011001110000000000
001100000000000000000011001110000000000
001101100000000000000111001100111110000
001111111100000000000111001111111111000
001110111110000000001110001111100111100
000000000111000000011100001110000001110
000000000111000000011000001110000001100
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Issues:

- The alphabet is no longer binary!
- Even if we express run lengths in binary, we still need an extra symbol for the comma!

Representing Lists

Generic Problem. Given only a binary alphabet, how can we express a *list* of numbers efficiently?

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$$m = \underbrace{\mathtt{000 \cdots 0}}_{m \text{ times}}\mathtt{1}$$

Sentinel `1` denotes the end of a number

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Question. How to address these shortcomings?

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Example. $m = 21$

- $21 = \texttt{10101}_2$ (binary)
- length $\ell = 4 = \texttt{0000}_1$
- encoding $21 \mapsto \texttt{00010101}$

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Goal. A *prefix code* for long runs (of 0s or 1s)

Two approaches.

- Represent lengths in *unary*
    - too long!
- Represent values in binary
    - not prefix code!

Idea. Combine the two approaches:

- Express m in binary (using $\log m$ bits)
- Write the length of m's binary representation (less 1) in unary
- Concatenate unary then binary parts

Example. $m = 21$

- $21 = 10101_2$ (binary)
- length $\ell = 4 = 0000_1$
- encoding $21 \mapsto 00010101$

Question. Why is this a prefix code?

Elias Encoding

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This encoding of positive integers is called the **Elias gamma code**.

Unique Decodability

Observation. If the Elias gamma code is a prefix code, then we should be able to unambiguously decode a sequence of concatenated encoded strings.

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PollEverywhere Question

What is the first value stored in the following encoded text:

`00001101000010111001111`

`pollev.com/comp526`

Run Length Encoding (RLE)

Encoding procedure. To compute the RLE of a binary source text *S*:

- Write the first bit of *S*.
- For each run, write the length of the run using the Elias gamma code

Example. Encode `11111100001111000000000`

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Example. Encode `111111000011100000000`

$\Longrightarrow$ `1 00110 010 010 0001001`

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Encoding procedure. To compute the RLE of a binary source text S:

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Decoding procedure. To decode an RLE encoded text C:

- Write the first bit b_0 of C
- Parse code word starting at index 1 of C and repeat b_0 that many times
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- Repeat until done

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Example. Decode `10011001001000001001`.

RLE Discussion

Generalizations and Applications.

- Can be extended to larger alphabets
    - write next character before run length
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Evaluation.

- Fairly simple and fast!
- Can compress n bits to $\Theta(\log n)$ bits (extreme best case!)
- Not good compression for many common datatypes
    - No compression for run lengths ≤ 6
    - Expansion for run lengths $k = 2, 6$.

Lempel-Ziv-Welch Encoding

Lempel-Ziv Compression

Compression so far: Exploit frequently repeated *single characters*

- Huffman: globally frequent characters (large alphabet)
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Observation. In many contexts, some substrings are much more frequent than others

- short words in English text (the, be, to, of, and, a, in, that)
- tags in HTML (`<div>`, `<a href`, ...)

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Lempel-Ziv covers a family of *adaptive* compression algorithms

- encode (frequently repeated) substrings of text with codewords
    - not just individual characters!
- Several variations of this idea
- Lempel-Ziv-Welch is a clean one (that is used in practice!)

LZW Idea

Codewords for different *strings* of text

- *Variable-to-fixed* encoding
  - all codewords have k bits (typical $k \approx 12$)
  - size of substring represented by each codeword varies
- Maintain a dictionary D (map) with 2^k entries
  - codewords are *values* in the dictionary
  - text strings are *keys* in the dictionary

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Encoding Idea.

- Initialize D with single characters Σ
- Start reading characters from S building up "words" (substrings) x
- If D contains x and next character is c, check if D contains xc
- If D does not contain xc, write $D(x)$ to C, **add xc to D**, and start building next word from c

Example

Consider $S =$ `N A N A S B A N A N A S`

code	string
0000	A
0001	B
0010	N
0011	S
0100	
0101	
0110	
0111	
1000	
1001	
1010	
1011	

C =

LZW in Pseudocode

```
procedure LZWEncode($S[0..n)$)
    $x \leftarrow \varepsilon$                          ▷ previous word, initially empty
    $C \leftarrow \varepsilon$                          ▷ output, initially empty
    $D \leftarrow$ all $c \in \Sigma_S$                 ▷ dictionary of codewords
    $k \leftarrow |\Sigma_S|$                           ▷ next free codeword
    for $i = 0, 1, \dots, n-1$ do
        $c \leftarrow S[i]$
        if $D$.ContainsKey($xc$) then
            $x \leftarrow xc$
        else
            $C \leftarrow C\,D$.Get($x$)                ▷ append codeword for $x$
            $D$.Put($xc, k$)
            $k \leftarrow k+1$, $x \leftarrow c$
        end if
    end for
    $C \leftarrow C\,D$.Get($x$)
end procedure
```

LZW in Pseudocode

```
procedure LZWENCODE($S[0..n)$)
    $x \leftarrow \varepsilon$                                  ▷ previous word, initially empty
    $C \leftarrow \varepsilon$                                  ▷ output, initially empty
    $D \leftarrow$ all $c \in \Sigma_S$                         ▷ dictionary of codewords
    $k \leftarrow |\Sigma_S|$                                   ▷ next free codeword
    for $i = 0, 1, \ldots, n-1$ do
        $c \leftarrow S[i]$
        if $D$.CONTAINSKEY($xc$) then
            $x \leftarrow xc$
        else
            $C \leftarrow C\, D$.GET($x$)                       ▷ append codeword for $x$
            $D$.PUT($xc, k$)
            $k \leftarrow k + 1$, $x \leftarrow c$
        end if
    end for
    $C \leftarrow C\, D$.GET($x$)
end procedure
```

For next time. Given C and D, how to decompress?

Next Time

Decompression

- Decoding LZW Encoding
- Making Texts Compressible

Scratch Notes