Lecture 13: Data Compression I

COMP526: Efficient Algorithms

Will Rosenbaum
University of Liverpool

Updated: November 14, 2024

Announcements

1. Programming Assignment 2 posted soon
2. Quiz 5 due Friday
   - Covers string matching
   - 2 questions (multiple choice)
   - Usual rules apply
3. Attendance Code:

508682

Meeting Goals

Discuss data compression!

- Introduce the data compression task
- Define character encoding and related terminology
- Define prefix codes
- Construct Huffman codes
- Prove optimality of Huffmann codes

Data Compression

The Story So Far

Emphasis. How do we process data?

- Data structures
  - How can we organize data perform primitive operations efficiently?
- Fundamental operations on arbitrary data:
  - sorting
  - string matching

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A New Question. How do we *store* and *transmit* data efficiently?

New Topics. Fundamental problems

1. Data Compression (starting today)
   - how to store data using as little space as possible
2. Error Correction (following topic)
   - how to automatically detect and correct errors in our data

The Data Compression Task

Σ_S source alphabet

Terminology.

- **source text:** string $S \in \boxed{\Sigma_S^*}$ to be stored/transmitted
    - Σ_S is some alphabet, e.g., Roman alphabet
- **coded text:** encoded data $C \in \Sigma_C^*$ that is actually stored/transmitted
    - typically have $\Sigma_C = \{0, 1\}$

Σ_C coded alphabet

computer alphabet, typical $\{0, 1\}$

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  - $E : \Sigma_S^* \to \Sigma_C^*$
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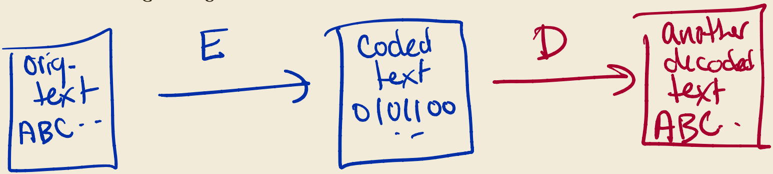

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Goal. Represent S using as little **space** as possible.

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Lossy vs. Lossless Compression.

- **Lossless Compression.** decoding recovers original text: $\boxed{D(E(S)) = S}$
  - Examples: zip (general archive), flac (audio), tiff (image)

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Our Focus: lossless compression!

The Quality of an Encoding Scheme

Goals of Encoding

- Efficiency of encoding/decoding ✓
- resilience to errors/noise in transmission
- security (encryption)
- integrity (detect modifications)
- size

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Our focus. Minimize the **size** of the encoded text.

- *data compression*

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Measure of quality. The *compression ratio:*

$$\frac{|C| \cdot \log|\Sigma_C|}{\big|S \cdot \log|\Sigma_S|\big|} \overset{\Sigma_C=\{0,1\}}{=} \frac{|C|}{|S| \cdot \log|\Sigma_S|}$$

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Question. Why all of the $\log|\Sigma|$s?

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← encoded text

Measure of quality. The *compression ratio:*

char encoded ↘ ← # bits to encode each char

$$\frac{|C| \cdot \log|\Sigma_C|}{|S| \cdot \log|\Sigma_S|} \overset{\Sigma_C=\{0,1\}}{=} \frac{|C|}{|S| \cdot \log|\Sigma_S|}$$

↖ (length)(bits per char for original text.)

Question. Why all of the $\log|\Sigma|$s?

- $\lceil \log \sigma \rceil$ is the minimum number of bits needed to represent σ distinct values (in binary)
- there are 2^b distinct binary strings of length b

$\sigma \leq 2^b$ to rep σ dist. chars. w/ b bits

$\Rightarrow \log \sigma \leq b$

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Interpretation. Compression ratios:

$< 1 \implies$ compression
- smaller values are better

$= 1 \implies$ no compression

$> 1 \implies$ encoded text is larger(?!)
- this is sometimes unavoidable …foreshadowing to next week

Data Compression Roadmap

Questions. When, how, and how much can we compress?

- **Part I:** Exploiting non-uniform character frequencies
  - Huffman Codes
- **Interlude:** Limits of data compression
- **Part II:** Exploiting repetition in texts
  - Run-length encoding
  - Lempel-Ziv-Welch (LZW) encoding
- **Part III:** Creating repetition in texts
  - Move-to-front transform
  - Burrows-Wheeler transform

Character Encoding

Question. How do computers encode~~d~~ English language text?

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Historical answer. ASCII use 7 bits per character

- all characters treated equally
- $2^7 = 128$ possible characters

$b_7 b_6 b_5$ →					0 0 0	0 0 1	0 1 0	0 1 1	1 0 0	1 0 1	1 1 0	1 1 1
Bits b_4	b_3	b_2	b_1	Column → Row ↓	0	1	2	3	4	5	6	7
0	0	0	0	0	NUL	DLE	SP	0	@	P	`	p
0	0	0	1	1	SOH	DC1	!	1	A	Q	a	q
0	0	1	0	2	STX	DC2	"	2	B	R	b	r
0	0	1	1	3	ETX	DC3	#	3	C	S	c	s
0	1	0	0	4	EOT	DC4	$	4	D	T	d	t
0	1	0	1	5	ENQ	NAK	%	5	E	U	e	u
0	1	1	0	6	ACK	SYN	&	6	F	V	f	v
0	1	1	1	7	BEL	ETB	´	7	G	W	g	w
1	0	0	0	8	BS	CAN	(	8	H	X	h	x
1	0	0	1	9	HT	EM	)	9	I	Y	i	y
1	0	1	0	10	LF	SUB	*	:	J	Z	j	z
1	0	1	1	11	VT	ESC	+	;	K	[	k	{
1	1	0	0	12	FF	FS	,	<	L	\	l	¦
1	1	0	1	13	CR	GS	—	=	M	]	m	}
1	1	1	0	14	SO	RS	.	>	N	^	n	~
1	1	1	1	15	SI	US	/	?	O	_	o	DEL

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Fixed Length

- all characters treated equally
- $2^7 = 128$ possible characters

Modern answer. Unicode

- $\sim 150{,}000$ representable characters (different scripts, emoji, etc.)
- several encoding schemes character $\rightarrow$ bits
- different characters' representations can have different lengths
  - e.g., ASCII characters represented by 8 bits

Diff lengths for diff chars.

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Character Encoding. Encode each character individually $E: \boxed{\Sigma_S} \to \boxed{\Sigma_C^*}$

- typically, $\underline{|\Sigma_S|} \gg |\Sigma_C|\ (= 2)$, so need several bits per character
- for $c \in \Sigma_S$, call $E(c)$ the **codeword** of c
- to encode a text, encode individual characters and concatenate

$E(A) = 0001$ AAB $\mapsto$ 00010001

$E(B) = 0010$ 0010 . . .

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Fixed vs. Variable Length Encoding

- *fixed length encoding* $\Longrightarrow$ all codewords have the same length (e.g. ASCII)
- *variable length encoding* $\Longrightarrow$ different lengths for different codewords (e.g. Unicode)

Fixed Length Codes

Advantages of fixed length codes

- fast decoding
  - use a lookup-table
  - can be as fast as a single array access
- local encoding
  - if character length is B, ith character starts at index $i \cdot B$

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Example. For (8-bit) ASCII encoding, how many (Roman alphabet) characters is this text? Where are the character divisions?

`01110100|01100101|01111000|01110100`

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`011101000110010101111000001110100`

Disadvantages of fixed length codes

- Inflexible (non-extensible)
  - how can we represent this awesome new emoji???
- Space inefficient
  - infrequently used characters require as much space as common characters
  - common characters are longer than they need to be

Variable Length Codes

Variable Length Advantages:

- more flexibility
- compressibility?

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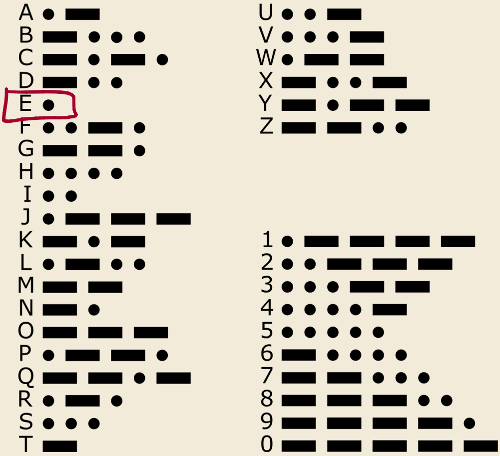
An old idea. Morse Code

- encode characters as "dots" and "dashes"
- more common characters are shorter



Variable Length Codes

Variable Length Advantages:

- more flexibility
- compressibility?

An old idea. Morse Code

- encode characters as "dots" and "dashes"
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Question. How many characters in the Morse code encoding?

3 letters in Σ_c

• ~~[illegible]~~ (pause)

A	• ▬
B	▬ • • •
C	▬ • ▬ •
D	▬ • •
E	•
F	• • ▬ •
G	▬ ▬ •
H	• • • •
I	• •
J	• ▬ ▬ ▬
K	▬ • ▬
L	• ▬ • •
M	▬ ▬
N	▬ •
O	▬ ▬ ▬
P	• ▬ ▬ •
Q	▬ ▬ • ▬
R	• ▬ •
S	• • •
T	▬
U	• • ▬
V	• • • ▬
W	• ▬ ▬
X	▬ • • ▬
Y	▬ • ▬ ▬
Z	▬ ▬ • •
1	• ▬ ▬ ▬ ▬
2	• • ▬ ▬ ▬
3	• • • ▬ ▬
4	• • • • ▬
5	• • • • •
6	▬ • • • •
7	▬ ▬ • • •
8	▬ ▬ ▬ • •
9	▬ ▬ ▬ ▬ •
0	▬ ▬ ▬ ▬ ▬

Codes Misbehaving

Banana

Bass

100

NA S

BANANA

BASS

PollEverywhere

Consider the following code

c	a	n	b	s
$E(c)$	0	10	110	100

What is the original text corresponding to the encoded text `1100100100`?

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Codes Misbehaving

Question. What was the issue with this code?

n 10

s 100

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Codes Misbehaving

Question. What was the issue with this code?

- The *relationship* between $E(n) = \texttt{10}$ and $E(s) = \texttt{100}$
  - If we read `10` in the encoded text, are we done reading a character?

PollEverywhere

Consider the following code

c	`a`	`n`	`b`	`s`
$E(c)$	`0`	`10`	`110`	`100`

What is the original text corresponding to the encoded text `1100100100`?

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Codes Misbehaving

Question. What was the issue with this code?

- The *relationship* between $E(n) = 10$ and $E(s) = 100$
  - If we read 10 in the encoded text, are we done reading a character?
- "Reasonable" codes should avoid this ambiguity!
  - We should *always* know when we're done reading a character.

PollEverywhere

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Prefix Codes and Tries

Definition. A character encoding E is a **prefix code** if no codeword $E(c)$ is a *prefix* of another code

Example.

c	A	E	N	O	T	␣
$E(c)$	01	101	001	100	11	000

code word $\boxed{01101}$

Can't have

01101 ???

as another codeword

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lookup table

Example.

c	A	E	N	O	T	␣
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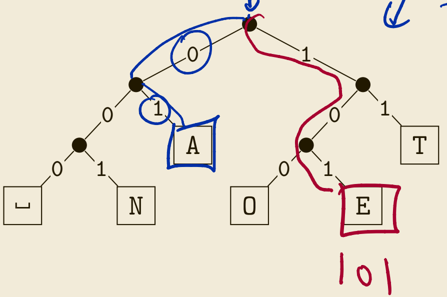
associati trie

Representation of prefix codes: the **trie** data structure!

- binary tree
- one leaf for each character
- edges labeled 0 or 1
- codewords = paths to leaves



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Prefix property ≡ all chars at leaves in trie

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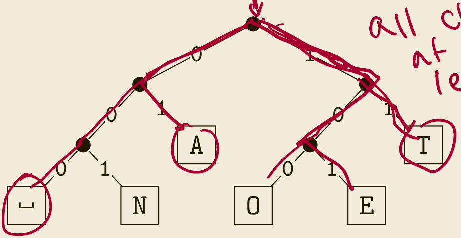

Encoding. Use the *table:* AN␣ANT ⟶ 010010000100111

Decoding. Use the *trie:* 111000001010111

T O ␣ E A T

Trie it Yourself

PollEverywhere Question

What is the result of using the trie on the right to decode the message:
`11001001001111`

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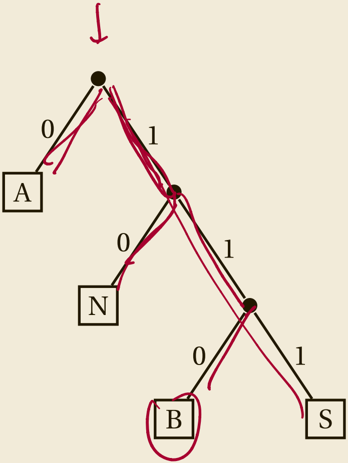

BANANAS

Fixed, Static, Adaptive

Note. In order to use a prefix code, we must also store the codewords!

- **fixed coding** uses the same code for all strings
  - e.g. ASCII, Unicode encodings (UTF-8)
- **static coding** uses the same codeword for each instance of a character in a text
  - codewords may different for different texts
  - must store/transmit the codewords as well as the encoded text!
- **adaptive coding** may change the codewords as the text is processed
  - codewords are stored implicitly within the coded message

Huffman Codes

Variable Length and Compression

Question. How can variable length encoding help with compression?

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Example. Consider the text `AAAAAAAAAAGGGH!`

- $\Sigma = \{\mathtt{A}, \mathtt{G}, \mathtt{H}, \mathtt{!}\}$
- Fixed length encoding:

c	A	G	H	!
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$\Longrightarrow$ Total encoded length = 30 (15 chars at 2 bits per char)

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- Exploiting frequency of `A` and `G`

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$\implies$ Total encoded length = $\boxed{22}$

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$\Longrightarrow$ Total encoded length = 22

Question. How can we find the **best possible** prefix code for compression?

Exploiting Character Frequency

Generic Optimization Problem. Suppose we are given

- a string S over the alphabet Σ;
- weights $\boxed{w(c) \geq 0}$ for each $c \in \Sigma$.

encoded length of c

Find the prefix code E for Σ that minimizes $\boxed{\sum_c w(c)\,|E(c)|}$

weight of character

Intuitively:

w(c) large, want to make |E(c)| smaller

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Example Weights. Take $w(c)$ to be the number of occurrences of c in S.

- note that $\sum_c w(c)\,|E(c)| = \boxed{|E(S)|}$ ← want as small as possible
- so solving optimization problem gives the shortest possible (prefix code) encoding of S!

times c appears

length of c encoded

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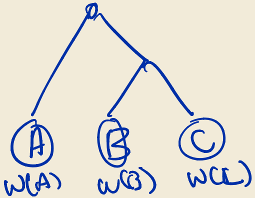
- I suppose we can with brute force: check all prefix codes
  - runs in exponential time in $|\Sigma|$
- Can we solve it *efficiently?*

Huffman Coding: Greed is Good

Idea. Build the character trie greedily from the leaves up.

- Prefix codes are binary trees with leaves labeled by Σ



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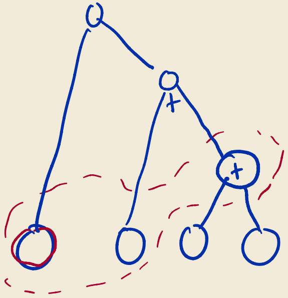
  2. the weight $w(c)$
- While $|A| > 1$:
  1. u and v are two lightest vertices
  2. add parent p to u and v
  3. set $w(p) = w(u) + w(v)$
  4. add p to A, remove u, v



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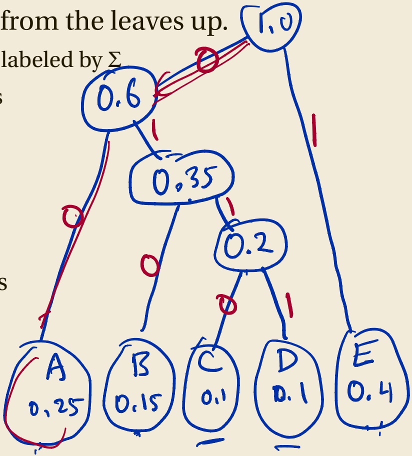
  3. set $w(p) = w(u) + w(v)$
  4. add p to A, remove u, v



00 010 0110 0111 1

Example.

- $\Sigma = \{A, B, C, D, E\}$
- weights $= \{0.25, 0.15, 0.1, 0.1, 0.4\}$

LOSSLESS Example

Example. Find the Huffman encoding for the text LOSSLESS.

weights: letter frequencies:

c	L	O	S	E
$w(c)$	2	1	4	1

L : 00
O : 010
S : 1
E : 011

$\downarrow$

$\boxed{00010110001111}$

LOSSLESS Example

Example. Find the Huffman encoding for the text `LOSSLESS`.

Three Steps:

1. Compute frequency counts $w(c)$ ✓
2. Build Huffman tree ✓
3. Write Huffman code from the tree ✓
(can build lookup table as well).

Huffman Analysis: Greed Works

Theorem

Given alphabet Σ *and weight function* $w : \Sigma \to \mathbf{R}_{\geq 0}$, *the Huffman coding schemes gives the minimum weighted codeword length* $\ell(E) = \sum_{c \in \Sigma} w(c) \cdot |E(c)|$ *among all prefix codes.*

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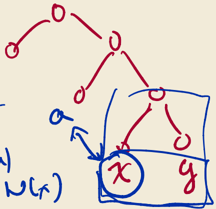
Opt trie

Proof sketch. Induction on $|\Sigma|$

Ind. hyp: on alphabets of size $|\Sigma|-1$, theorem holds

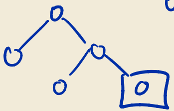
- Let E^* be an optimal encoding/trie
- Claim: $\exists$ sibling leaves x, y at max depth
- Swap x and y for two min weight leaves, a, b
- Optimal code for $\Sigma' = \Sigma \setminus \{a, b\} \cup \{\boxed{ab}\}$ gives optimal code for Σ (verify this!)
- By inductive hypothesis, Huffman gives optimal code for Σ'
- So we get an optimal code for Σ $\square$

missing



$w(a) < w(x)$

Swapping can only decrease $\Sigma\, w(c)|E(c)|$



← merged ab

Huffman Computational Efficiency

Question. For an alphabet of size $m = |\Sigma|$ and weights w, how efficiently can we build the Huffman code?

m alphabet size

- Maintain a collection A of active vertices
- Initially A is set of leaves, labeled with
  1. a character $c \in \Sigma$
  2. the weight $w(c)$

$O(m)$

- While $|A| > 1$:
  1. u and v are two lightest vertices
  2. add parent p to u and v
  3. set $w(p) = w(u) + w(v)$
  4. add p to A, remove u, v

$O(1)$

- Construct the codeword table

Priority Queue
$\hookrightarrow$ Heap

$O(m \log m)$ to build
+ log m per removal

$\Rightarrow O(m \log m)$ time

Tie Breaking Rules

So far we have two ambiguities in our Huffman trie description:

1. Which child is right/left child of the parent?
2. What do we do if weights are tied?

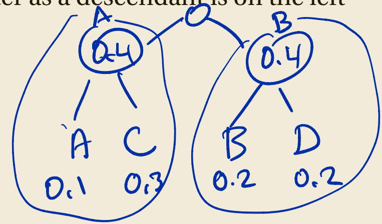
Tie Breaking Rules

So far we have two ambiguities in our Huffman trie description:

1. Which child is right/left child of the parent?
2. What do we do if weights are tied?

Conventions.

- Smaller weight child is on the left
- All ties broken by earliest character in alphabetical order
    - for internal vertices, the one containing the alphabetically first character as a descendant is on the left



Huffman and Entropy

A Thought Experiment

Suppose I have an alphabet $\Sigma = \{c_1, c_2, \ldots, c_n\}$ and I choose a character c_i *at random* to transmit

- each c_i is chosen with probability p_i.

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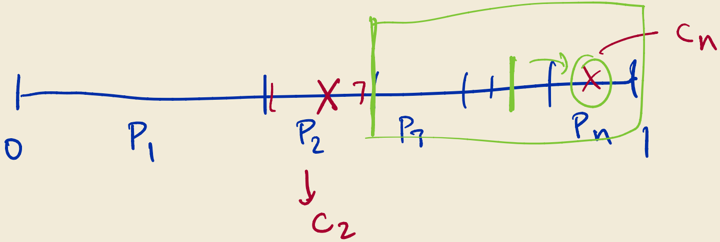
$P_1 + P_2 + \cdots + P_n = 1$

Idea. Think of p_i as sub-intervals of $[0, 1]$.

- Outcome is a random point x in $[0, 1]$
- c_i corresponds to the interval containing x
- Use binary search to find the interval!



A Thought Experiment

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- Outcome is a random point x in $[0,1]$
- c_i corresponds to the interval containing x
- Use binary search to find the interval!
- If the interval has width $\boxed{p_i}$ need $\boxed{\log(1/p_i)}$ queries to determine interval
- The *expected* (average) number of queries is then

prob of picking $x \in$ interval i

$$\underline{\mathcal{H}(p_1, p_2, \ldots, p_n)} = \sum_{i=1}^{n} p_i \log\left(\frac{1}{p_i}\right)$$

bits transmitted

- $\mathcal{H}$ is the **entropy** of the distribution over Σ

$0 \le \quad \le \log n \quad (p_i = \frac{1}{n})$

Properties of Entropy

Setup. We choose elements from $\Sigma = \{c_1, c_2, \ldots, c_n\}$ randomly, each c_i chosen with probability p_i.

One can show:

- Entropy $\mathscr{H}$ is a *lower bound* on the average number of bits needed to transmit a random character from Σ

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One can show:

- Entropy $\mathcal{H}$ is a *lower bound* on the average number of bits needed to transmit a random character from Σ
- If we use a Huffman encoding of $\boxed{\Sigma}$ $c_1, \ldots, c_n$
  - weights $\boxed{w(c_i) = p_i}$
  - transmit the Huffman codeword $E(c_i)$

  Then the average length $\ell(E)$ of the transmitted word satisfies

$$\boxed{\mathcal{H}} \leq \ell(E) \leq \mathcal{H} + 1$$

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One can show:

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  - transmit the Huffman codeword $E(c_i)$

  Then the average length $\ell(E)$ of the transmitted word satisfies

$$\mathcal{H} \leq \ell(E) \leq \mathcal{H} + 1$$

Conclusion. Huffman coding gives (nearly) the best possible *average* compression for *randomly* generated texts!

Emprical Entropy

Definitions. For a fixed string S over alphabet $\Sigma = \{c_1, c_2, \ldots, c_\sigma\}$, we define the **relative frequency** of character c_i in S to be

$$\boxed{p_i} = \frac{\#\text{ occurrances of } c_i \text{ in } S}{|S|}$$

The **empirical entropy** of S is then

$$\mathcal{H}_0(S) = \mathcal{H}(p_1, p_2, \ldots, p_\sigma).$$

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The length of the Huffman encoded text $C = E(S)$ is

$$\underline{|C|} = \sum_{i=1}^{\sigma} |S|_{a_i} |E(c_i)| = n \sum_{i=1}^{n} p_i |E(c_i)| = \boxed{n\ell(E).}$$

Applying the previous slide gives $\boxed{\mathcal{H}_0(S)n} \le |C| \le \boxed{(\mathcal{H}_0(S)+1)}n$.

- Entropy and Huffman coding length are intimately connected

Next Time

More Compression!

- Limits of Compressibility
- Compressing Repetitive Texts

Scratch Notes