Lecture 11: String Matching II

COMP526: Efficient Algorithms

Updated: November 7, 2024

Will Rosenbaum
University of Liverpool

289105

Announcements

1. **NO QUIZ THIS WEEK!**
2. Programming Assignment Posted
    - **TESTING CODE UPDATED**
        - small bug in tritonic array generation
        - download new version
    - Due Wednesday, 13 November
3. Attendance Code:

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Meeting Goals

Discuss String Matching procedures:

- Brute Force
- DFA procedure
- Knuth-Morris-Pratt

String Matching

The String Matching Problem

Σ = alphabet

Input:

- A **text** $T \in \Sigma^*$ of length n
- A **pattern** $P \in \Sigma^*$ of length m (typically $m \ll n$)

Output:

- The index of the **first occurrence** of P in T, or -1 if T does not contain P as a substring:
  - $\min\{i \mid T[i, i+m) = P\}$

Example.

0 1 2 3 4 5 6

- $T = 101100\boxed{1101}1101$
- $P_1 = 1101$
  - Output: $i \leftarrow 6$
- $P_2 = 000$
  - Output: $i \leftarrow -1$

Brute Force Matching

Brute Force Matching

Guess an index i where a match might occur

- Possible guesses $i = 0, 1, \ldots, n - m - 1$

Check if match at i:

- is $T[i, i + m) = P$?
- verify each character individually

Cost = number of comparisons made

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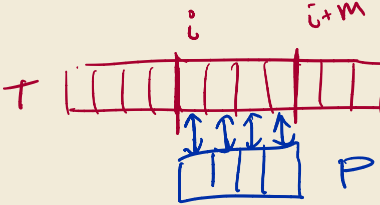


```
procedure VerifyMatch(T, P, i)
    j ← 0
    while j < m do
        if T[i + j] ≠ P[j] then
            return False
        end if
        j ← j + 1
    end while
    return True
end procedure
```

Cost = number of comparisons made

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end procedure
```

Cost = number of comparisons made

Parameters:
n = size of T
m size of P

PollEverywhere Question

What are the worst case and best case running times of VerifyMatch?

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Brute Force Matching

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Check if match at *i*:

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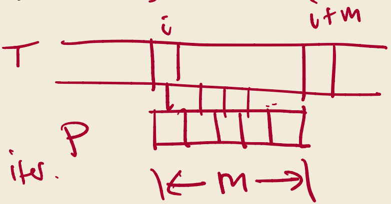
procedure VerifyMatch(T, P, i)
    $j \leftarrow 0$
    while $j < m$ **do**
        if $T[i+j] \neq P[j]$ **then**
            return False
        end if
        $j \leftarrow j+1$
    end while
    return True
end procedure

$\leq m$ iter.

Cost = number of comparisons made

Best and Worst Cases:

Worst :



$\Theta(m)$ = Worst

Best $O(1)$

$\rightarrow T[i] \neq P[0]$

Brute Force Matching

Guess an index i where a match might occur

- Possible guesses $i = 0, 1, \ldots, n - m - 1$

Check if match at i:

- is $T[i, i + m) = P$?
- verify each character individually

Cost = number of comparisons made

Brute force. Guess and check every value $i = 0, 1, \ldots, n - m - 1$

- Worst case running time is $\Theta(nm)$
  - What is example has cost $\Omega(nm)$?
- Best case cost is $\Theta(m)$

T = aaaaaa...a

P = aaa..ab

aa..ab

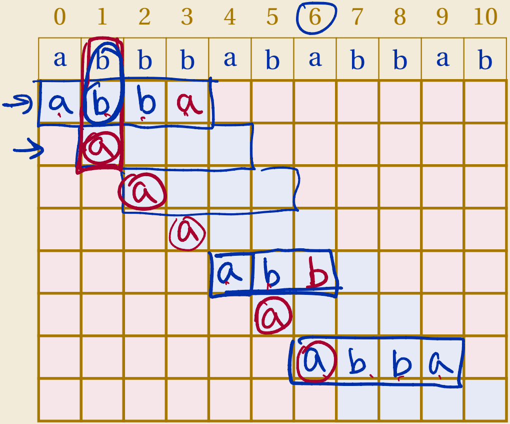
every char in T is compared to every char in P

find match at index $i = 0$

Brute Force Example

Example

- $T = abbbababbab$
- $P = abba$



procedure BRUTEFORCEMATCH(T, P)
    for $i = 0, 1, \ldots, n - m - 1$ **do**
        if VERIFYMATCH(T, P, i) **then**
            return i
        end if
    end for
    return -1
end procedure

← return 6

Brute Force Efficiency

The **worst case** complexity of brute force search is $\Theta(nm)$...

...but when is this **actually** achieved?

$T = a\,a\,a\,a\,a \cdots a$

$P = a\,a\,a \cdots a\,b$

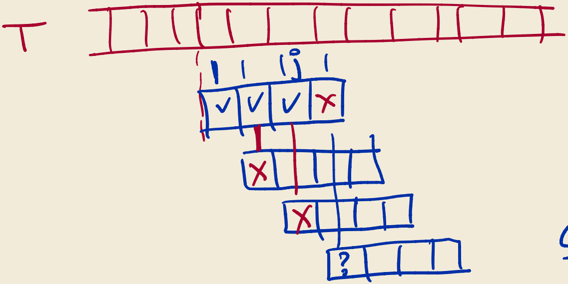
Brute Force Efficiency

The **worst case** complexity of brute force search is $\Theta(nm)$…

…but when is this **actually** achieved?

Example. Consider the case where P contains *no repeated characters.*

$P = P_1 P_2 \cdots P_m \neq$



In this case never do $> 2n$ comparisons

$\leq$ 2n comparisons

Brute Force Efficiency

The **worst case** complexity of brute force search is $\Theta(nm)\ldots$

$\ldots$ but when is this **actually** achieved?

Example. Consider the case where P contains *no repeated characters*.

- Claim: brute force search running time is now $O(n)$
  - In fact, at most $2n$ comparisons made!
  - *Why?*

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- Claim: brute force search running time is now $O(n)$
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  - *Why?*
- Which of these comparisons were unnecessary?
  - How can you search with fewer comparisons?

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  - *Why?*
- Which of these comparisons were unnecessary?
  - How can you search with fewer comparisons?

More generally: How can we use results of *previous comparisons* to avoid making unnecessary comparisons in the future?

- Goal: never re-read a character from T!

Matching with a DFA

Sliding Comparisons

Example

- `T = aababab babacaa`
- `P = ababaca`

$$a \quad a \quad b \quad a \quad b \quad a \quad b \quad b \quad a \quad b \quad a \quad b \quad a \quad c \quad a \quad a$$

Idea:

- Scan through T keeping track of current matches
- Each new character T read, compare it to next character of P
- If mismatch slide P so that **longest prefix** of P matches

Sliding Comparisons

Example

- T = aababbabbabacaa
- P = abababaca

$$
\begin{array}{lccccccccccccccc}
T & a & a & b & a & b & a & b & b & a & b & a & b & a & c & a & a \\
P & a & b & \cancel{a} & \cancel{b} & \cancel{a} & \cancel{c} & \cancel{a} & & & & & & & & &
\end{array}
$$

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Sliding Comparisons

Example

- T = aabababbabacaa
- P = ababaca

don't need

	a	a	b	a	b	a	b	b	a	b	a	b	a	c	a	a
	a	b	a	b	a	c	a	← do these								
→		a	b	a	b	a	c	a								
				✓	✓	✓	b									

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Sliding Comparisons

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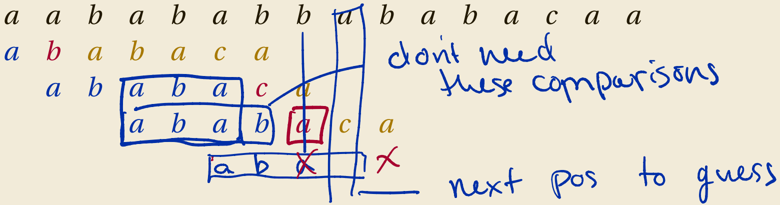

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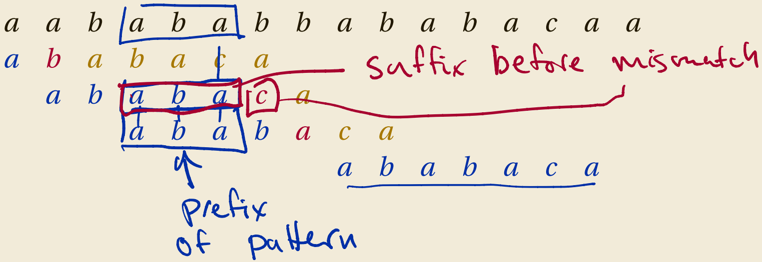

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Representing States and Matches

Question. What information do we need to compute and store to determine next comparison?

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- How many matches in P have we made so far?
- What what is the longest matching prefix for each possible next character in T
  - if we read character x, how far do we need to “slide” P to match a prefix?

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Information to store

- **states** that represent number of matches with current prefix of P
- **transitions** from current state to next states, depending on next character read from T

Note. This information depends *only* on the pattern P, not the text T.

DFAs

A **Deterministic Finite Automaton** (**DFA**) consists of:

- A finite set Q of **states**
- A finite **alphabet** Σ
- A **transition function** $\delta : Q \times \Sigma \to Q$
- An **initial state** $q_0 \in Q$
- A set $F \subseteq Q$ of **accepting states**

rule: if in state q and read character c (in Σ) which state to move to

$$\to 0 \xrightarrow{a} 1 \xrightarrow{b} 2 \xrightarrow{a} 3 \xrightarrow{b} 4$$

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Interpretation. A DFA is used to determine if a string (text) T has some property (e.g., containing a pattern P):

- Start at the state q_0
- Read characters from T sequentially
  - if in state q and read character c, move to state $\underline{\delta(q, \sigma)}$

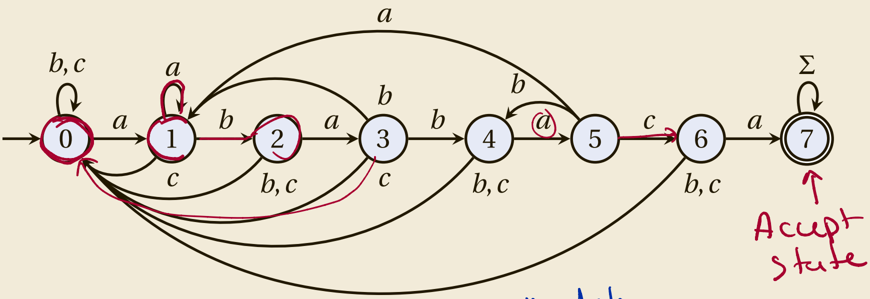
transition function

- Return TRUE if end in "accepting" state

DFA Example

Example

- T = aabacaababacaa
- P = ababaca



match

text	*a*	*a*	*b*	*a*	*c*	*a*	*a*	*b*	*a*	*b*	*a*	*c*	*a*	*a*
state	1	1	2	3	0	1	1	2	3	4	5	6	7	7

DFA Efficiency

PollEverywhere Question

Given a DFA for matching $P[0,m)$ in $T[0,n)$, what is the running time of applying the DFA? Assume following links is $O(1)$ time.

1. $\Theta(nm)$
2. $\Theta(n \log m)$
3. $\Theta(n+m)$
4. $\boxed{\Theta(n)}$

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DFA Efficiency

Observe: If we are *given* a DFA, executing it

- reads each character of T once
- updates state once per character

$\Longrightarrow$ running time $O(n)$

So the overall running time for pattern matching with a DFA is $O(n) +$ time to build DFA

- assuming computation of δ is $O(1)$.

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But how do we build the DFA? ???

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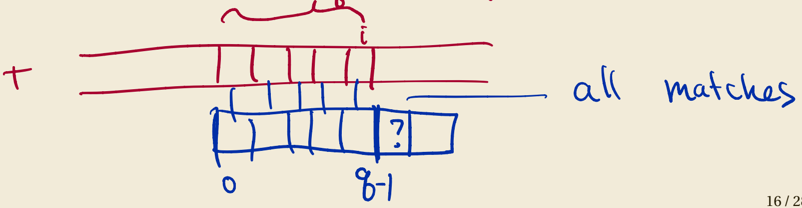
DFA Interpretation & Construction

Semantic Question. What does it *mean* to be in state q?

In state q after reading $T[i]$

$\iff$

Chars in T from $i-q+1$ to i $=$ $P[0..q]$



DFA Interpretation & Construction

Semantic Question. What does it *mean* to be in state q?

- Current position in T matches P up to the first q characters
- Symbolically $T[j-q+1, j] = P[0, q)$

Question. What happens when we read $T[j+1]$?

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- If $T[j+1] = P[q]$, transition to state $q+1$
- If $T[j+1] \neq P[q]$, find the length $q' \leq q$ of the longest prefix of P that matches $T[j-q', j+1]$ that matches $P[0, q')$

	a	a	b	a	b	a	b	b	a	b	a	b	a	c	a	a
		0	1	2	3	4	5									
$q = 5$		a	b	a	b	a	c	a								
$q' = 4$				a	b	a	b	a	c	a						

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	a	a	b	a	b	a	b	b	a	b	a	b	a	c	a	a
$q=5$		a	b	a	b	a	c	a								
$q'=4$				a	b	a	b	a	c	a						

- **Insight:** if $T[j+1] = c$ this is the same as matching $P[0..q]$ against $P[1..q)c$
  - we can use the DFA constructed so far to find this!

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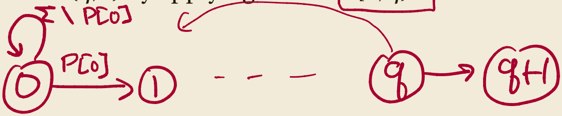
Inductive Construction.

- Start with states 0 and 1 with

$$\delta(0, c) = \begin{cases} 1 & \text{if } P[0] = c \\ 0 & \text{otherwise.} \end{cases}$$

- Once we've constructed DFA up to state q:
  - take $\delta(q, P[q]) = q + 1$
  - for $c \neq P[q]$, find $\delta(q, c)$ by applying DFA to $\boxed{P[1, q)c}$



DFA Interpretation & Construction

Todo: Fix img!

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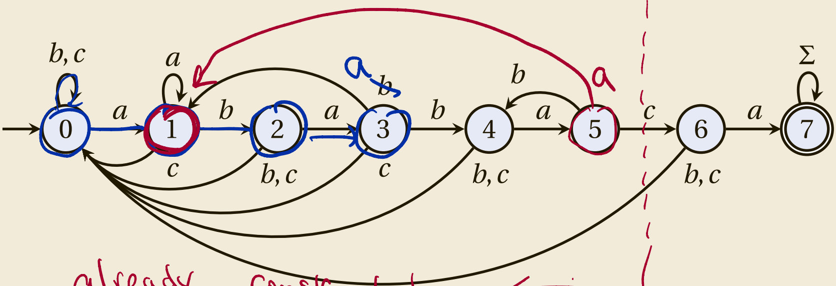
~~a~~baba$\boxed{a}$

Example. Compute $\delta(5, a)$ for $P =$ `ababaca`.



already constructed ←

DFA Interpretation & Construction

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Analysis (idea).

- Argue by induction on q that the DFA enters state q on reading $T[j]$ if and only if q is the largest number such that $\boxed{T[j-q+1,j] = P[0,q).}$

DFA Lookup Table Construction

DFA *diagrams* are great for humans, but not so great for computers…

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Problems.

1. How do we represent the DFA in a computer friendly format?
2. How do construct the DFA in that format efficiently?

DFA Lookup Table Construction

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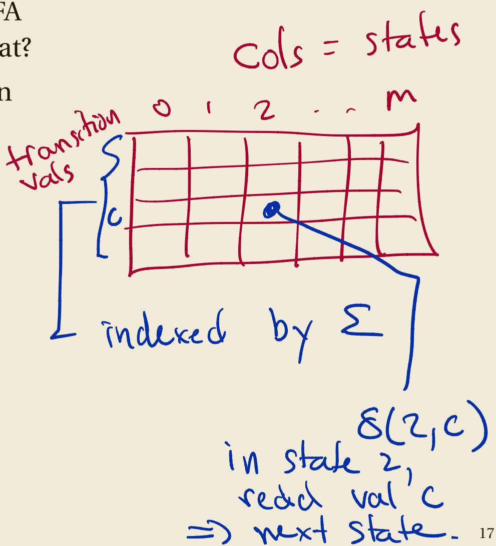
Solutions.

1. Store a **lookup table** $\delta[\,][\,]$
   - columns = states, rows = characters
   - $\delta[q][c] \leftarrow \delta(q, c)$



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   - columns = states, rows = characters
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2. Compute column by column
   - trick: keep track of state for $P[1, q)$ because we'll reuse this for each $P[1, q)c$

DFA Lookup Table Construction

Solutions.

1. Store a **lookup table** $\delta[\,][\,]$
   - columns = states, rows = characters
   - $\delta[q][c] \leftarrow \delta(q, c)$
2. Compute column by column
   - trick: keep track of state for $P[1, q)$ because we'll reuse this for each $P[1, q)c$
   - x is largest value with $\boxed{P[0, x)} = \boxed{P[q - x, q]}$

procedure CONSTRUCTDFA($P[0..m)$)
    for $c \in \Sigma$ **do**
        $\delta[0][c] \leftarrow 0$
    end for
    $\delta[0][P[0]] \leftarrow 1$
    $x \leftarrow 0$
    for $q = 1, 2, \ldots, m - 1$ **do**
        for $c \in \Sigma$ **do**
            $\delta[q][c] \leftarrow \delta[x][c]$
        end for
        $\delta[q][P[q]] \leftarrow q + 1$
        $x \leftarrow \delta[x][P[q]]$
    end for
end procedure

init first col

iterate over columns

DFA Lookup Table Construction

Example. $P[0..6) =$ `ababaca`

$\delta(c,q)$	0	1	2	3	4	5	6
$P[q]$	a	b	a	b	a	c	a
a	1	1	3				
b	0	2	0	. . .			
c	0	0	0				
x	0	0	1				

procedure ConstructDFA($P[0..m)$)
  for $c \in \Sigma$ **do**
    $\delta[0][c] \leftarrow 0$
  end for
  $\delta[0][P[0]] \leftarrow 1$
  $x \leftarrow 0$
  for $q = 1, 2, \ldots, m-1$ **do**
    for $c \in \Sigma$ **do**
      $\boxed{\delta[q][c] \leftarrow \delta[x][c]}$
    end for
    $\boxed{\delta[q][P[q]] \leftarrow q+1}$
    $x \leftarrow \delta[x][P[q]]$
  end for
end procedure

copy col x
to col q

DFA Lookup Table Construction

PollEverywhere Question

What is the running time of CONSTRUCTDFA when P has length m and $|\Sigma| = s$?

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procedure CONSTRUCTDFA($P[0..m)$)
  for $c \in \Sigma$ **do**
    $\delta[0][c] \leftarrow 0$
  end for
  $\delta[0][P[0]] \leftarrow 1$
  $x \leftarrow 0$
  for $q = 1, 2, \ldots, m-1$ **do**
    for $c \in \Sigma$ **do**
      $\delta[q][c] \leftarrow \delta[x][c]$
    end for
    $\delta[q][P[q]] \leftarrow q + 1$
    $x \leftarrow \delta[x][P[q]]$
  end for
end procedure

$\Theta(s)$

$\Theta(1)$

$\Theta(1)$

s

$O(1)$

$m-2$ iter. $\longrightarrow \Theta(ms)$

DFA Lookup Table Application

Pitting it Together

- Construct the DFA
- Apply the DFA

```
procedure ApplyDFA(T[0..n], δ, m)
    q ← 0
    for i = 0, 1, ..., n − 1 do
        q ← δ[q][T[i]]
        if q = m then
            return i
        end if
    end for
    return −1
end procedure
procedure DFAMatch(P[0..m], T[0..n])
    δ ← ConstructDFA(P, T)
    return ApplyDFA(T, δ, m)
end procedure
```

start at state 0

O(n) time

DFA Lookup Table Application

Pitting it Together

- Construct the DFA
- Apply the DFA
- Running time is $\Theta(n + m|\Sigma|)$
  - $\Theta(m|\Sigma|)$ for making DFA
  - $\Theta(n)$ for applying DFA
- Additional space overhead: $\Theta(m|\Sigma|)$
  - store the DFA

could be a lot of space

```
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```

Knuth-Morris-Pratt

Failure Link Automaton

DFA efficiency.

- Space/time to build DFA: $\Theta(m|\Sigma|)$
- Time to execute DFA: $\Theta(n)$

$\implies$ Overall time is $\Theta(n + m|\Sigma|)$

  - additional space overhead is $\Theta(m|\Sigma|)$

Question. Can we perform string matching in time $O(n)$ with *less space overhead?*

Failure Link Automaton

DFA efficiency.

- Space/time to build DFA: $\Theta(m|\Sigma|)$
- Time to execute DFA: $\Theta(n)$

$\Longrightarrow$ Overall time is $\Theta(n + m|\Sigma|)$

  - additional space overhead is $\Theta(m|\Sigma|)$

Question. Can we perform string matching in time $O(n)$ with *less space overhead?*

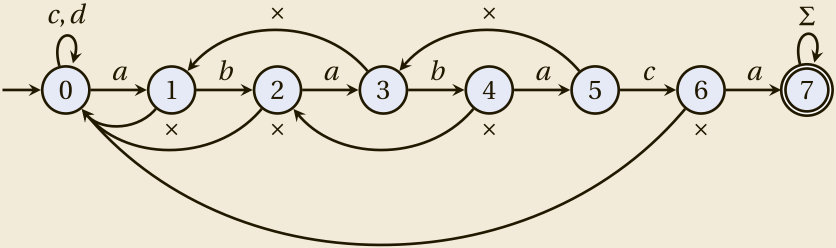
Idea. When comparison fails, don't have a separate transition for each failing character

- Just record failure and "shift" pattern as far forward as possible

Failure Link Automaton

Example

- T = aabacaababacaa
- P = ababaca



text	a	a	b	a	c	a	a	b	a	b	a	c	a	a
states														

FLA Execution

A **Failure Link Automaton** (**FLA**) consists of:

- A finite set Q of **states**
- A finite **alphabet** Σ
- A **transition function** $\varphi : Q \times (\Sigma \cup \{\times\}) \to Q$
- An **initial state** $q_0 \in Q$
- A set $F \subseteq Q$ of **accepting states**

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Execution. To apply and FLA to T

- Start at the state q_0
- Read characters from T sequentially
  - if in state q and read character c:
    - if $\varphi(q, c)$ is defined, move to state $\varphi(q, c)$
    - otherwise move to state $\varphi(q, \times)$ and **re-read** c
- Return TRUE if end in "accepting" state

FLA Execution

PollEverywhere Question

Given an FLA for a pattern P of length m, how many times could we follow failure links for a single character c read from T in the worst case?

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FLA Running Time

More careful analysis

- If we match up to $P[j]$, then we can only follow up to j back links
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Amortized cost of each character read from T

- If read character c is a **match**:
  - pay 1 for comparison
  - put 1 unit cost in the **bank**
- If read character c is a **mismatch**
  - *withdraw* 1 from the bank
- By analysis above account balance is always non-negative

$\Longrightarrow$ amortized cost of each comparison is 2

$\Longrightarrow$ hence overall running time of execution is $O(n)$

FLA Construction

Observation. Each state q has

- 1 forward link to state $q+1$
- 1 fail link

Given P, we don't need to store forward link label:

- forward link label from $P[q] = P[q+1]$

Only need to store fail link state!

- this can be stored as a single array of size m

$\Longrightarrow$ only $O(m)$ space overhead

FLA Construction

Definition. The **failure link array** *fail* of P the array of m numbers that stores the (index of) the next state for each failure

- How do we construct it?

FLA Construction

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- How do we construct it?
- Again x is length of largest prefix that matches a suffix of $P[1, q)$

Example. $P[0..6)$ = `ababaca`

q	0	1	2	3	4	5	6
$P[q]$	a	b	a	b	a	c	a
$fail[q]$							

```
procedure FailureLink(P[0, m))
    fail[0] ← 0
    x ← 0
    for j = 1, 2, ..., m − 1 do
        fail[j] ← x
        while P[x] ≠ P[j] do
            if x = 0 then
                x ← −1
                break
            else
                x ← fail[x]
            end if
        end while
        x ← x + 1
    end for
end procedure
```

FLA Construction

Question. What is the running time of FAILURELINK on input of size m?

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        end while
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FLA Construction

Question. What is the running time of FAILURELINK on input of size m?

Observations.

- x incremented once per j
- $fail[x] < x$
- Each "while" iteration decrements x

So at most $2m$ updates to x

- cf. amortized analysis
- x = bank balance

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- Scan along $T[0,n)$
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- Maintain position in $P[0,m)$
  - index j
  - current prefix match
- When $T[i] = P[j]$, increment i and j
- Otherwise, $j \leftarrow fail[j]$
  - unless $j = 0$, then $i \leftarrow i + 1$

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- Otherwise, $j \leftarrow fail[j]$
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```
procedure KMP(T[0..n), P[0..m))
    fail ← FailureLink(P)
    i ← 0
    j ← 0
    while i < n do
        if T[i] = P[q] then
            i ← i + 1, j ← j + 1
            if j = m then return i − j
        else
            if j ≥ 1 then
                j ← fail[j]
            else
                i ← i + 1
            end if
        end if
    end while
end procedure
```

KMP Algorithm

Analysis:

- Running time $O(n+m)$
  - $O(m)$ to build *fail*
  - $O(n)$ to apply KMP
  - analysis uses **amortized analysis**
- Additional space $O(m)$
  - just need to store *fail* and indices

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Clean Takeaway:

fail$[j]$ is the length of the longest prefix of $P[0..j]$ that is a suffix of $P[1..j]$

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DFA vs FLA

Question. Which is better? DFA matching or KMP algorithm?

- KMP has overall running time $O(n+m)$
  - amortized 2 comparisons per T access
- DFA has overall running time $O(n+m|\Sigma|)$
  - 1 comparison per T access
  - $|\Sigma|$ dependence

Next Time

More String Matching!

Scratch Notes