# Lecture 5: Data Structures II

## COMP526: Efficient Algorithms

Updated: October 17, 2024

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University of Liverpool

# Announcements

1. Second Quiz Open, due Friday 11:59 pm
    - Similar format to before
    - Covers asymptotic (Big-O) notation
    - Quiz is **closed resource**
        - No books, notes, internet, etc.
        - Do not discuss until after submission deadline (Friday night, after midnight)
2. CampusWire —
    - Use for discussion of material, questions about lectures, etc
    - Public comments for matters related to module content & administration
    - https://campuswire.com/p/GBB00CD7A, Code: 4796
3. Attendance Code:

160521

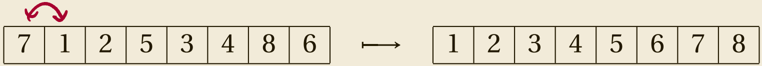
# Meeting Goals

- Introduce Programming Assignment 1: Prefix Reversal Sorting
- Discuss the Queue ADT and implementations
- Introduce the Priority Queue ADT
- Introduce the heap data structure

# Programming Assignment 1

# Non-Standard Sorting

**Fundamental Task:** sorting a list of elements from smallest to largest



| 7 | 1 | 2 | 5 | 3 | 4 | 8 | 6 |
|---|---|---|---|---|---|---|---|

$\longmapsto$

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|

Typical basic (unit cost) operations:

- **compare** two elements to see which is larger
- **swap** two elements in the array

# Non-Standard Sorting

**Fundamental Task:** sorting a list of elements from smallest to largest

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 7 & 1 & 2 & 5 & 3 & 4 & 8 & 6 \\ \hline\end{array} \longmapsto \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline\end{array}$$

Typical basic (unit cost) operations:

- **compare** two elements to see which is larger
- **swap** two elements in the array

**Non-standard** sorting models:

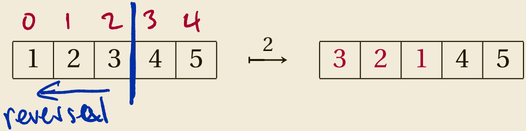
- natural in contexts other than sorting arrays
- e.g., sorting physical objects with physical constraints
- compare and swap may not be elementary operations

Credit: Andy Goldsworthy

# Sorting with Prefix Reversals

**Basic Operation:** Prefix Reversal

- Reverse the elements up to index *i* in a list/array
- For example



- Natural operation for
  - DNA
  - stacks of physical objects

**Basic Algorithmic Question:** Given an array $a$ of length $n$, what is the fewest number of **prefix reversal** operations necessary to sort $a$?

# Your Task

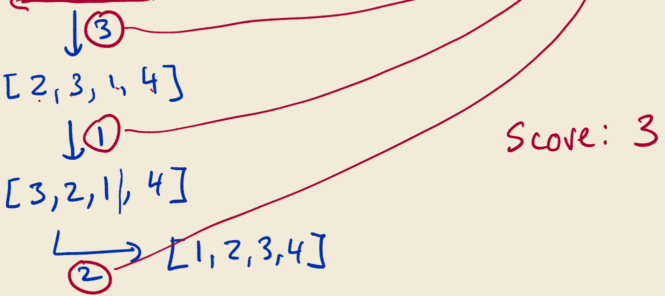
**Input:** an array (`list`) *a* of numbers between 1 and *n*.

**Output:** the array *p* of *prefix reversals* that when applied to *a* will result in a sorted array.

**Goal:** sort each array *a* using the fewest possible prefix reversals.

0 1 2 3

**Example:** Sort `[4, 1, 3, 2]`



3

[2, 3, 1, 4]

1

[3, 2, 1, 4]

2

[1, 2, 3, 4]

Output: [3, 1, 2]

Score: 3

# Your Task

**Input:** an array (`list`) *a* of numbers between 1 and *n*.
**Output:** the array *p* of *prefix reversals* that when applied to *a* will result in a sorted array.
**Goal:** sort each array *a* using the fewest possible prefix reversals.
**Example:** Sort `[4, 1, 3, 2]`

4 3 5 | 6 1 2 ) 2

5 3 4 6 1 2 || ) 5

2 1 6 4 | 3 5 ) 3

* 4 6 1 2 3 5 *

## PollEverywhere Question

Starting from the array
0 1 2 3 4 5
`[4, 3, 5, 6, 1, 2]`
what is the resulting array after performing the following prefix reversals?
`[2, 5, 3]`

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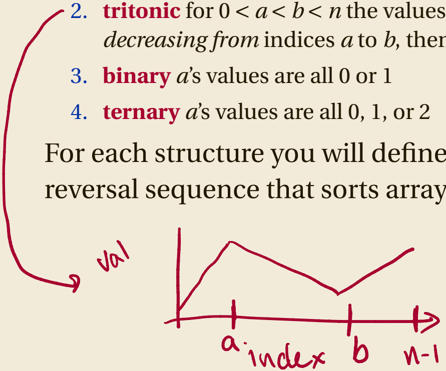
# More Specifically

**Array Structures** for input

1. **random permutation** $a$ is uniformly random shuffling of numbers from 1 to $n$
2. **tritonic** for $0 < a < b < n$ the values of $a$ are *increasing* from indices 0 to $a$, *decreasing from* indices $a$ to $b$, then *increasing* from indices $b$ to $n-1$.
3. **binary** $a$'s values are all 0 or 1
4. **ternary** $a$'s values are all 0, 1, or 2

For each structure you will define a function that generates a prefix reversal sequence that sorts arrays with the given structure.



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**Scoring**:

- your program must correctly sort all arrays
- points for minimizing the number of prefix reversals over all challenge arrays

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**Suggestion:** it is possible to sort any array of length $n$ with fewer than $2n$ prefix reversals

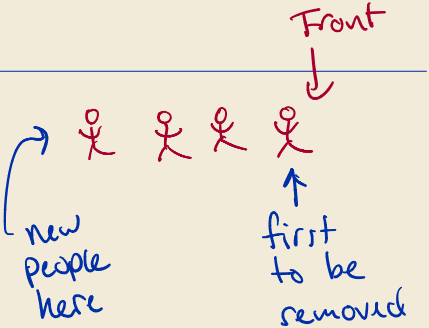
- start by implementing a simple baseline procedure to sort all arrays

# Queues

# From Last Time

- Stack ADT — Abstract Data Type
    - **linked list** implementation
    - **array** implementation
- Amortized analysis

# The Queue ADT



### Queues, Intuitively

Goal: to store a *collection* of elements

- elements arranged as in a queue at Tesco
- new people enter the **back** of the queue
- only the person at the **front** of the queue can be removed (serviced)

First In, First Out (FIFO) priority

# The Queue ADT

concatenation

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## Queues, Formally

- $S$ is the state of the queue, initially $S = \varnothing$
- $S.\textsc{Enqueue}(x) : \boxed{S} \mapsto \boxed{xS}$
- $\underline{S.\textsc{Front}}() :$ returns $x_{n-1}$ where $S = x_0 x_1 \cdots x_{n-1}$
- $S.\textsc{Dequeue}() : \underline{Sx} \mapsto \underline{S}$, returns $\underline{x}$
- $S.\textsc{Empty}()$ returns $\textsc{True} \iff S = \varnothing$

# The Queue ADT

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First In, First Out (FIFO) priority

**Tons of Applications!**

- Scheduling
- Messaging
- …

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# List Backed Queues

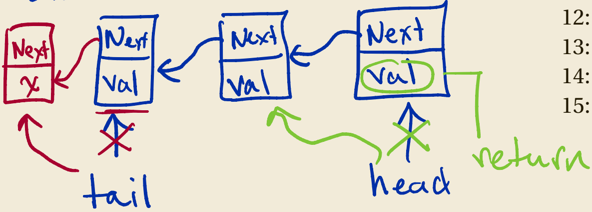
**Idea**

- Store each element in a NODE
- Store references to NODE:
    - head at the front of the queue
    - tail at the back of the queue

```
class LISTQUEUE
    NODE head
    NODE tail
    procedure ENQUEUE(x)
        $n \leftarrow$ new NODE
        $n$.data $\leftarrow x$
        tail.next $\leftarrow n$
        tail $\leftarrow n$
    end procedure
    procedure DEQUEUE
        $n \leftarrow$ head
        head $\leftarrow n$.next
        return $n$.data
    end procedure
end class
```

Node

State of Queue



# List Backed Queues

**Idea**

- Store each element in a NODE
- Store references to NODE:
  - head at the front of the queue
  - tail at the back of the queue

**Issues:**

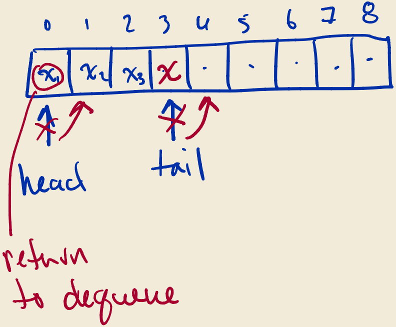
- Similar to linked list stack implementation
  - Locality of reference
  - NODE memory overhead

```
class ListQueue
    Node head
    Node tail
    procedure Enqueue(x)
        $n \leftarrow$ new Node
        $n$.data $\leftarrow x$
        tail.next $\leftarrow n$
        tail $\leftarrow n$
    end procedure
    procedure Dequeue
        $n \leftarrow$ head
        head $\leftarrow n$.next
        return $n$.data
    end procedure
end class
```

# Array Backed Queues

**Idea:**

- Store elements in the stack in an array
- Maintain indices of head and tail



*Ignores resizing/checking if full*

```
class ArrayQueue
    $a \leftarrow$ new array, size $n$
    head, tail $\leftarrow 0$
    procedure Enqueue(x)
        $a$[tail] $\leftarrow x$
        tail $\leftarrow$ tail + 1
    end procedure
    procedure Dequque
        head $\leftarrow$ head + 1
        return $a$[head − 1]
    end procedure
end class
```

# Array Backed Queues

**Idea:**

- Store elements in the stack in an array
- Maintain indices of head and tail

**What is the problem here?**

- What happens if we repeatedly call
  - ENQUEUE(1)
  - DEQUEUE()
  - ENQUEUE(1)
  - DEQUEUE()
  - …

*Ignores resizing/checking if full*

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class ArrayQueue
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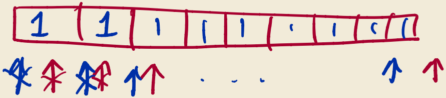

# Array Backed Queues

**Idea:**

- Store elements in the stack in an array
- Maintain indices of head and tail

**The fix:**

- Use *circular arrays*
- Perform index arithmetic *modulo n* (array size)
- All operations are then $O(1)$
    - amortized $O(1)$ time if resizing by doubling size

*Ignores resizing/checking if full*

```
class ArrayQueue
    $a \leftarrow$ new array, size $n$
    head, tail $\leftarrow 0$
    procedure Enqueue(x)
        $a[\text{tail}] \leftarrow x$
        tail $\leftarrow$ tail $+ 1$ $\underline{\text{mod } n}$
    end procedure
    procedure Dequque
        head $\leftarrow$ head $+ 1$ $\underline{\text{mod } n}$
        return $a[\text{head} - 1 \underline{\text{ mod } n}]$
    end procedure
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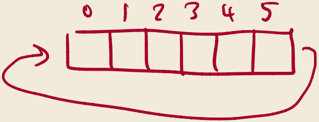

# Priority Queues

# The (Min) Priority Queue ADT

## Priority Queues, Intuitively

Goal: to store a *collection* of elements

- Each element $x$ has an associated *priority*, $\boxed{p(x)}$
- New elements **inserted** with prescribed priorities
- Can access/remove element with the *minimum* priority in the collection

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### Priority Queues, Formally

- $S$ is the state of the queue, initially $S = \varnothing$
- $S.\textsc{Insert}(\enclose{circle}{x}, p(x)) : S = \boxed{x_0 x_1 \cdots x_i x_{i+1} \cdots x_{n-1}} \mapsto x_0 x_1 \cdots \boxed{x_i\, x\, x_{i+1}} \cdots x_{n-1}$
  - where $\underline{p(x_i)} \le \underline{p(x)} < \underline{p(x_{i+1})}$
- $S.\textsc{Min}() :$ returns $x_0$ where $S = \boxed{x_0} x_1 \cdots x_{n-1}$
- $S.\textsc{RemoveMin}() : \underline{xS \mapsto S}$, returns $x$

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**Applications**

- efficient sorting
- implementing “greedy” algorithms
- resource management

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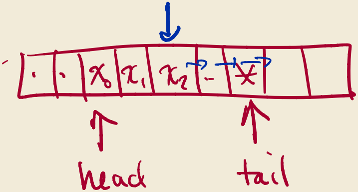
# Naive Priority Queue Implementations

**Array backed implementation**

- Store element/priority pairs, sorted by priority
- MIN and REMOVEMIN can be implemented in $\underline{O(1)}$ time
- INSERT is $\underline{\Theta(n)}$ worst-case
    - must **shift** elements around $x$



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**Linked list backed implementation**

- Store element/priority pairs, sorted by priority
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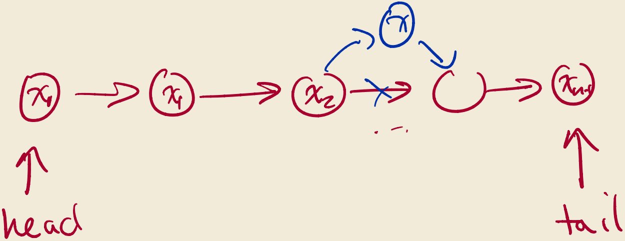
    - must **find** location to insert $x$



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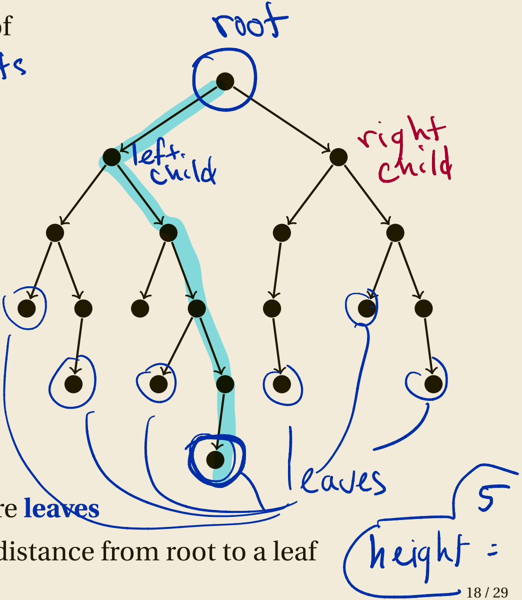
**Question.** Can we perform *all* operations in $o(n)$ time?

# Heaps

# Binary Trees

A (rooted) **binary tree** consists of

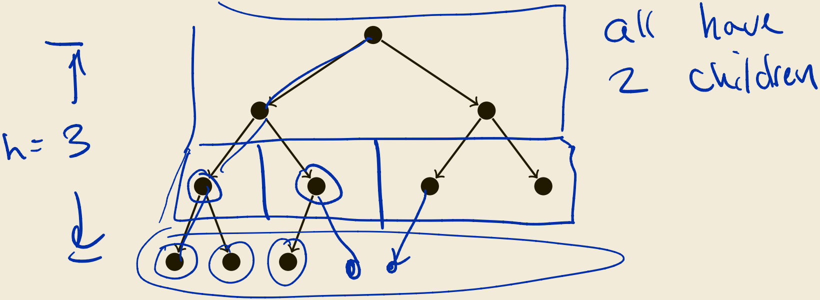
- A set $V$ of vertices — dots
- A distinguished vertex called the **root**
- Each vertex has (possibly empty):
  - left child
  - right child
- Non-root vertices have a parent
- All vertices are descendants of the root
- Vertices without children are **leaves**
- The **height** is the maximal distance from root to a leaf



# Complete Binary Trees

A **complete binary tree** of height $h$ is a binary tree in which:

- All vertices up to depth $h-2$ have exactly two children
- At most one vertex at depth $h-1$ has one child
  - if $v$ has children, then vertices to the left of $v$ have two children



# Properties of Complete Binary Trees

## Proposition

Suppose $T$ is a complete binary tree of height $h$. Then the number $n$ of vertices of $T$ satisfies $\boxed{2^h \leq n \leq 2^{h+1} - 1.}$

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dist from root

## Proof.

- The number of vertices at $\boxed{\textit{depth}}$ $d \le h - 1$ is $2^d$
  - prove by induction on $d$
- There for the total number vertices up to depth $h - 1$ is

$$n' = \underline{1} + \underline{2} + \underline{4} + \cdots + \underline{2^{h-1}} = \boxed{2^h - 1}$$

  - prove formula by induction
- At depth $\underline{h}$, the number of vertices is between $\underline{1 \text{ and } 2^h}$

Therefore, total $n$ is between $2^h = n' + 1$ and $2^{h+1} - 1 = \boxed{n' + 2^h.}$ ☐

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## PollEverywhere Question

If a complete binary tree $T$ has $n$ vertices, what is its height?

1. $h = \Theta(1)$
2. $h = \Theta(\log n)$
3. $h = \Theta(\sqrt{n})$
4. $h = \Theta(n)$

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Why?

$2^h \leq n$

$\log(2^h) \leq \log n$

$h \leq \log n \Rightarrow h = O(\log n)$

$n \leq 2^{h+1} \Rightarrow \log n \leq \log(2^{h+1})$

$\Rightarrow \log n \leq h+1$

$h \geq \log n - 1$

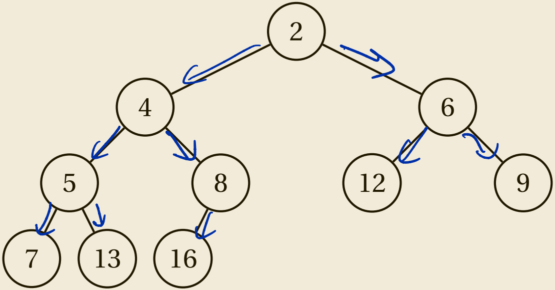
$h = \Omega(\log n)$

# Min Heaps

## Definition

A **heap** is a complete binary tree $T$ with the following properties:

- each vertex (node) has an associated value from an *ordered set*
    - for each pair of values $p$ and $q$ we can *compare* $p < q$
- the value associated with each vertex $v$ is *smaller* than the values associated with its children



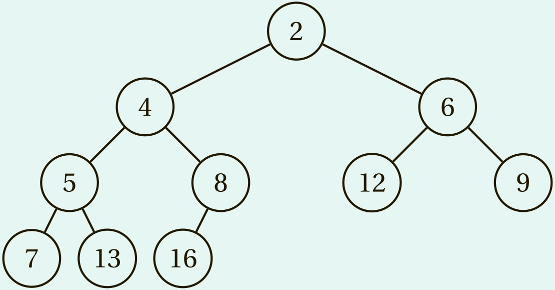
# Inserting Into a Heap

**Question**

Given a heap $T$, how can we *efficiently* **insert a new a value** into $T$ and maintain the heap properties?

**Example**

How to insert the value 3?



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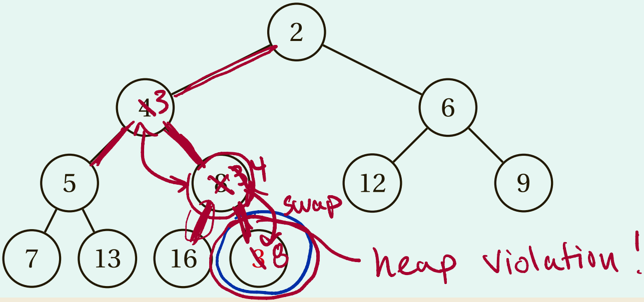

# “Bubble Up” Insert Procedure

```
procedure INSERT(p)
    $v \leftarrow$ new vertex storing $p$
    $u \leftarrow$ first vtx with $< 2$ children
    add $v$ as $u$'s child
    PARENT($v$) $\leftarrow u$
    while value($v$) $<$ value($u$) and $u \neq \perp$ do
        SWAP(value($v$), value($u$))
        $v \leftarrow u$
        $u \leftarrow$ PARENT($v$)
    end while
end procedure
```

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```

Stop if v is root

**PollEverywhere Question**

What is the running time of INSERT if $T$ has $n$ vertices?

1. $\Theta(1)$
2. $\Theta(\log n)$
3. $\Theta(\sqrt{n})$
4. $\Theta(n)$

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# "Bubble Up" Insert Procedure

Cheat: store this!

not obvious.

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```

$O(1)$

$O(1)$

$O(1)$

Why is running time $\Theta(\log n)$?

\# iterations $\leq$ height of tree

$= \Theta(\log n) \leftarrow$

**PollEverywhere Question**

What is the running time of INSERT if $T$ has $n$ vertices?

worst case

1. $\Theta(1)$
2. $\Theta(\log n)$
3. $\Theta(\sqrt{n})$
4. $\Theta(n)$

Best case running is $O(1)$.

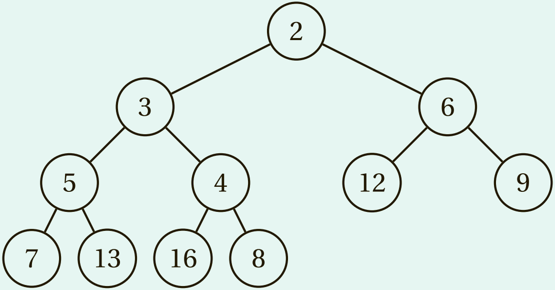
# Removing Min

## Question

Given a heap $T$, how can we *efficiently* **remove the minimum value** from $T$ and maintain the heap properties?

## Example

How to remove 2?



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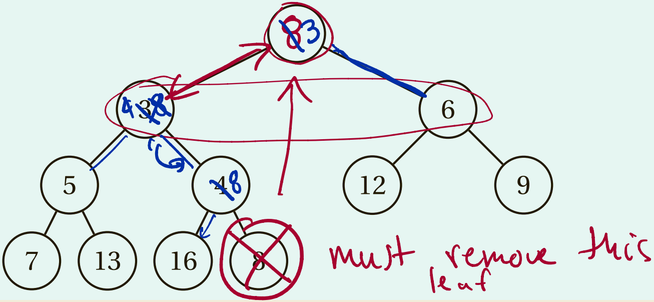
## Example

How to remove 2?



# “Trickle Down” Remove Min Procedure

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procedure RemoveMin
    $v \leftarrow$ tree root
    $w \leftarrow$ “last” vertex in tree
    value($v$) $\leftarrow$ value($w$)
    remove $w$ from tree
    $u \leftarrow$ smaller of $v$’s children
    while value($v$) $\geq$ value($u$) and $u \neq \perp$ do
        Swap(value($v$), value($u$))
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**PollEverywhere Question**

What is the running time of RemoveMin if $T$ has $n$ vertices?

1. $\Theta(1)$
2. $\Theta(\log n)$
3. $\Theta(\sqrt{n})$
4. $\Theta(n)$

pollev.com/comp526

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**Less Obvious Choice: Arrays!**

- **For next time:** think about how you could represent a heap using an **array** and minimal additional overhead!

# Heap Priority Queues

**So far** we heaps only store values from a ordered set

- A priority queue needs two data fields:
    1. a value $x$
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- The *priorities* are from an ordered set
- To implement a priority queue with a heap
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**The payoff**

- INSERT in time $\Theta(\log n)$
- REMOVEMIN in time $\Theta(\log n)$
- MIN in time $\Theta(1)$

# Next Time: More Trees!

- Searching Sorted Arrays
- Binary Search Trees
- Balanced Binary Trees

# Scratch Notes