Lecture 3: Machines and Models

COMP526: Efficient Algorithms

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Announcements

1. First quiz live now, due Friday
   - Administered through Canvas
   - One question, multiple choice
   - 20 minutes
   - Covers basic logic (Tuesday's lecture, this week's tutorial, posted notes)
   - **Don't start until you're ready to take the quiz.**
2. Programming Assignment 1 released next week
   - Due 13 November
3. Attendance Code: 825332

Meeting Goals

- Finish discussion of mathematical induction
- Analyze algorithm correctness with loop invariants
- Formalize the RAM model of computation
- Introduce asymptotic notation

Mathematical Induction

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Statement T or F for each nat. #

The Principle of Mathematical Induction

Let P be a predicate over the natural numbers $\mathbf{N} = \{0, 1, 2, \ldots\}$. Suppose P satisfies

- Base case: $P(0)$ is true.
- Inductive step: For every $i \in \mathbf{N}$, $P(i) \implies P(i+1)$.

Then for every $n \in \mathbf{N}$, $P(n)$ is true. In strictly symbolic notation:

$$(P(0)) \wedge (\forall i[P(i) \implies P(i+1)]) \implies \forall n P(n).$$

Loop Invariants

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Given an algorithm A containing a loop, a loop invariant is a predicate P on the iterations of the loop such that for each iteration i, $P(i)$ is satisfied at the end of the i-th iteration of the loop.

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An Uninteresting Example

Consider the following procedure

```
1: procedure COUNT(n)
2:     t ← 0
3:     for i = 1, ..., n do
4:         t ← t + 1
5:     end for
6:     return t
7: end procedure
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Loop Invariant

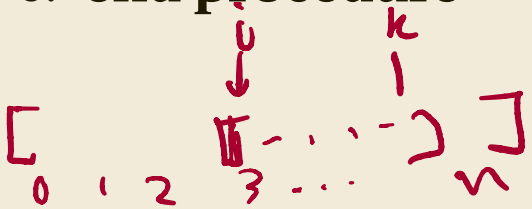
After iteration i, t stores the value i.

Induct on i

- Base case: t initialized to 0. (Line 2)
- Inductive step:
  - Suppose after iteration i, t stores the value i (inductive hypothesis)
  - Then in iteration $i+1$ after line 4, t stores the value $i+1$.

A More Interesting Example

```
1: procedure MinIndex((a, i, k))                    ▷
   Find the index of the minimum
   value stored in array a between
   indices i and k.
2:     m ← i
3:     for j = i, i + 1, . . . , k do
4:         if a[j] > a[m] then
5:             m ← j
6:         end if
7:     end for
8:     return m
9: end procedure
```



PollEverywhere Question

What loop invariant does the loop in MinIndex satisfy that will help us analyze its behavior?

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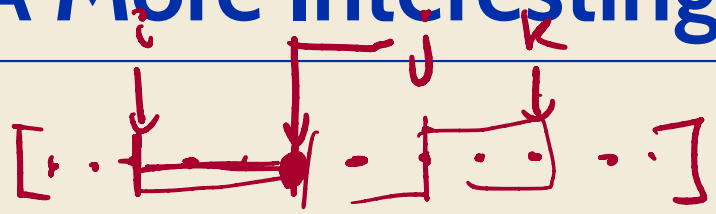


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After iteration j, m stores the index of the minimum value of a between indices i and j.

A More Interesting Example

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2: $m \leftarrow i$
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4: **if** $a[j] < a[m]$ **then**
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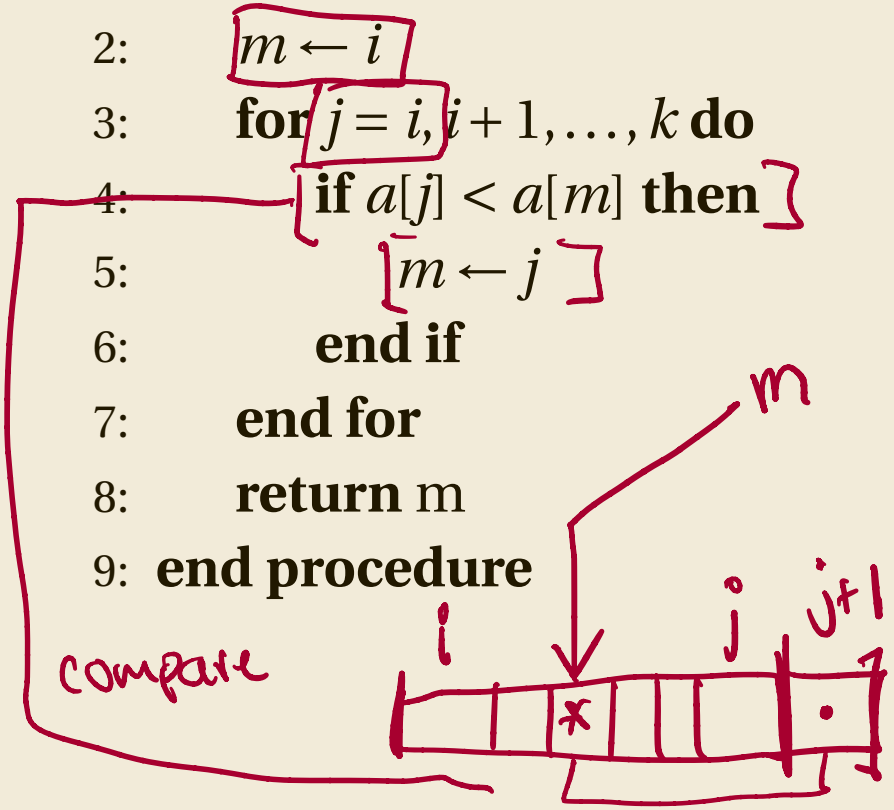

Loop Invariant

After iteration j, m stores the index of the minimum value of a between indices i and j.

Proof.

Induct on j

- Base case: $j = i$.
- Inductive step: $j \implies j+1$

□

Further Application

Consider the following algorithm that uses MININDEX as a subroutine:

```
1: procedure SELECTIONSORT(a, n)          ▷ Sort the array a of size n
2:     for i = 1, 2, ..., n do
3:         j ← MININDEX(a, i, n)
4:         SWAP(a, i, j)
5:     end for
6: end procedure
```

Exercise (Tutorials)

Show that SELECTIONSORT correctly sorts any array a of length n.
Specifically:

- Find a suitable loop invariant satisfied by SELECTIONSORT
- Prove your loop invariant holds (by induction)
- Argue that your loop invariant implies the final array is sorted

Induction and Recursion

Induction is essential in reasoning about *recursively defined* methods.

A Recursive Method

```
1: procedure MYSTERY(n)
2:     if n = 1 then
3:         return 1
4:     end if
5:     return 2n − 1 + MYSTERY(n − 1)
6: end procedure
```

Mystery(2)
= 2
2n-1 + Mystery(1)
⇒ 2·2-1+1 = 4

PollEverywhereQuestion

What is the output of MYSTERY(5)?

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Analysis of a Mystery

1: **procedure** MYSTERY(n)
2: **if** $n = 1$ **then**
3: **return** 1
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Claim

For all n, MYSTERY(n) returns the value n^2.

Proof.

Induction on n. Base Case: $n = 1$. line 3

Inductive step: Suppose MYSTERY(n) $= n^2$. Then

$$\begin{aligned} \text{MYSTERY}(n+1) &= 2n+1 \\ &\quad + \text{MYSTERY}(n) \\ &= 2n+1+n^2 \\ &= (n+1)^2. \end{aligned}$$

inductive hypothesis

□

Modelling Computation

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compute $x+y$ for any x and y

compute $3+7$

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input → processing → output

Example

The SELECTIONSORT algorithm

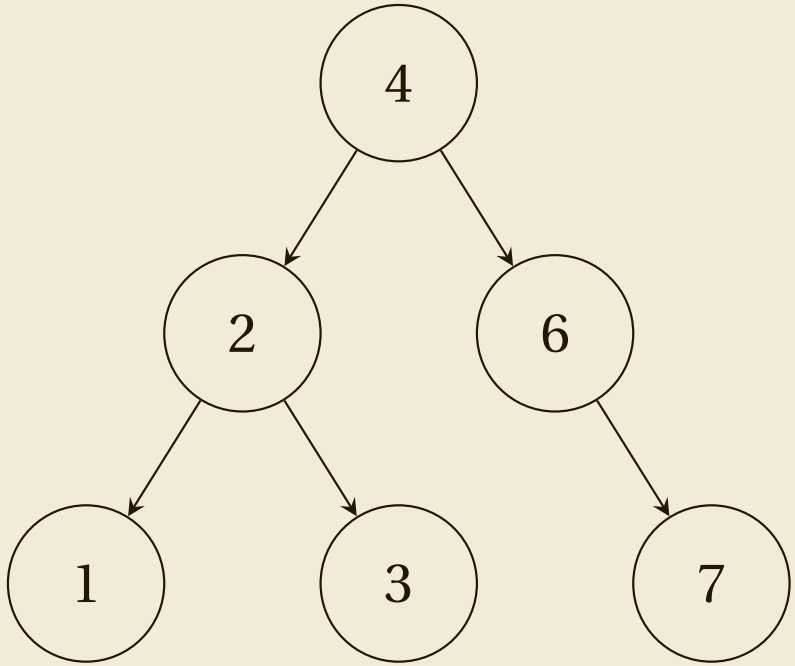
- sorts *any* array of size n
- elementary operations/logic

What is a Data Structure?

A **data structure** is

1. a rule for *encoding data* in computer memory, and
2. algorithms for accessing and manipulating the data according to the specified encoding.
    - See next week for much more detail!

Typical example: binary search trees



What Makes an Algorithm "Good?"

Overarching Goal: find the "best" algorithm and data structures for a task

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Algorithm analysis gives us a way to

- compare different algorithms
- predict their performance (efficiency) in applications

Limitations of Empirical Analysis

Question. Why not just implement your algorithm and test it on a real computer with real data?

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Our Focus is formal analysis of algorithms on an abstract computer

- *prove* results for our computational model
- results apply to any computer that satisfy our assumptions

Note. Neither formal nor empirical analysis is *better*—both are important to computer science!

- this module just focuses on formal analysis

Data Models

To perform formal algorithm analysis, we must specify

1. the computational model
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Types of performance:

- **worst-case performance**
  Over all possible inputs, what is the *worst* cost of our algorithm's execution?
- **best-case performance**
  Over all possible inputs, what is the *best* cost of our algorithm's execution?
- **average-case performance**
  What is the *average* or *expected* cost for a of a *random* input

Typically, we analyze performance as a function of **input size**, n

$\implies$ measure performance as a function of n: how does performance *scale* with input size

Computational Models

A **computational model** defines

1. *syntax:* what operations can be performed
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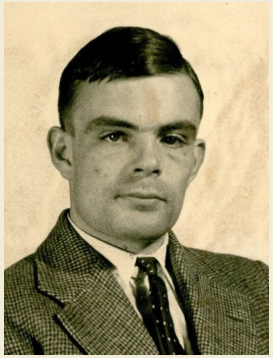
Successful & general computational models defined in the 1930's

Credit: Princeton University

Alonzo Church
(Lambda Calculus)



Credit: Unknown

Alan Turing
(Turing Machines)

Random Access Machines

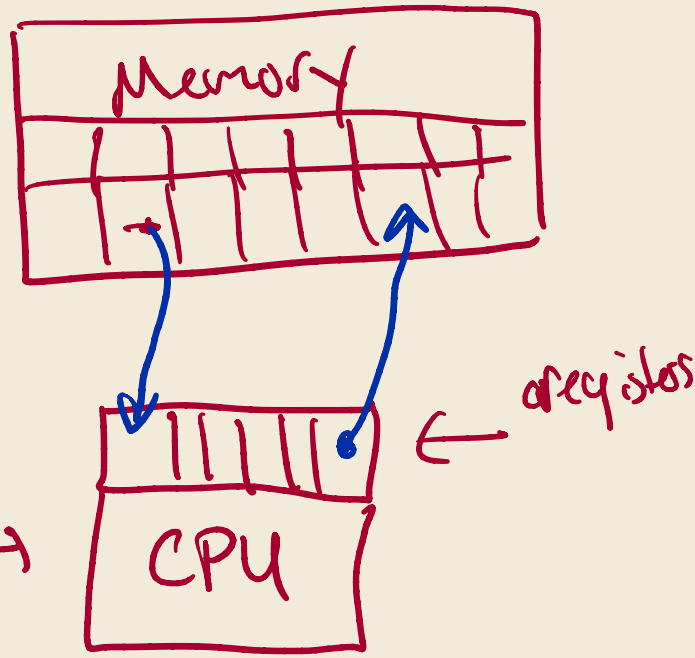
The RAM Model consists of:

- Unlimited **memory**, access by address
  - stores program, input, intermediate data
  - each address stores fixed sized "word"
- Fixed number of **registers**
  - stores "working" data
- Elementary **instructions**
  - load & store: move data between registers/memory
  - arithmetic operations: bit-wise ops, addition/subtraction (fixed precision),...
  - conditional/unconditional jump
- **cost** = number of instructions executed

RAM is a standard model for *sequential* computation, similar to assembly

Credit: LANL

Jon von Neumann



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Simplifying abstractions:

- Higher level abstract **pseudocode**:
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Pseudocode can (in principle) be implemented in RAM model, just as C++ can be compiled to assembly

procedure EUCLID(a, b)
  while $b \neq 0$ **do**
    $t \leftarrow b$
    $b \leftarrow a \bmod b$
    $a \leftarrow t$
  end while
  return a
end procedure

Greatest common ~~denominator~~ of a b

divisor

(Important) Things We Ignore

System & Hardware Level Details

- Memory allocation
  - required to implement arrays, dynamic memory usage, etc
- Pointers
  - correspondence between variable names, values, and memory addresses
- Support for procedures/methods/functions
  - call stack, call frame, etc.

This are all fundamental problems in computer science, just not within the purview of COMP526

- See modules on operating systems, compilers, programming languages, etc.

Asymptotic Notation

Measuring (Time) Efficiency

Recall. We measure *(time) efficiency* in terms of number of elementary operations performed

- assume all operations are unit cost

We want a robust measure of efficiency that is *independent* of particular "real world" costs of operations

- focus on how the number of operations scales with input size

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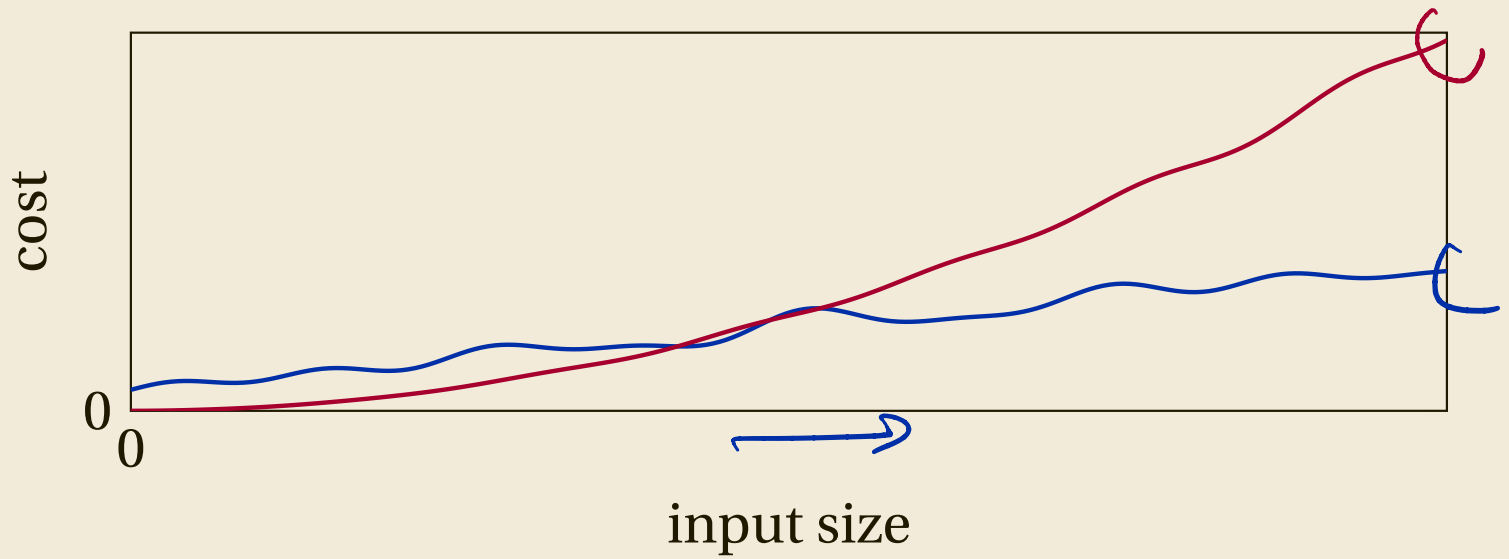
This motivates the study of **asymptotic analysis**.



Big-O Notation

Goal of "Big-O" or asymptotic notation: a way of describing the *growth* of functions that is:

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natural #s (input size)

cost

Definition

Suppose f and g are functions from $\mathbf{N}$ to $\mathbf{R}^+$. Then we say that $f = O(g)$ (read: f is *big* O of g) if there exist constants $N_0 \in \mathbf{N}$ and $C \in \mathbf{R}$ such that for all $n \in \mathbf{N}$

$$n \geq N_0 \Longrightarrow f(n) \leq Cg(n).$$

Equivalently, $f = O(g) \iff \limsup \frac{f(n)}{g(n)} < \infty$

Big-O in Pictures

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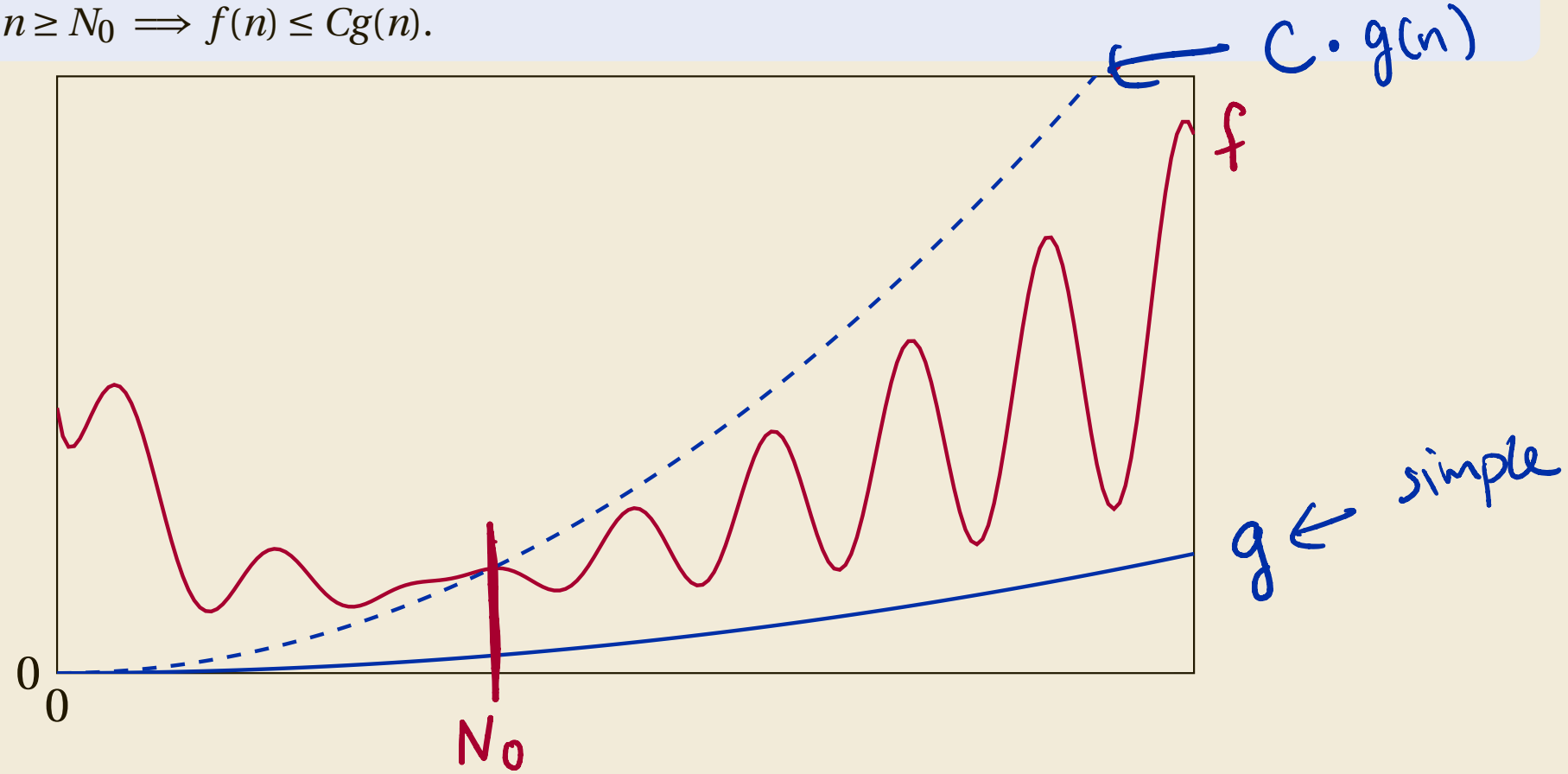


Properties of O

Proposition

Suppose $f, f_1, f_2, g, g_1, g_2, h$ are functions and a is any constant. Then:

1. $(\forall n f(n) \le a) \implies f = O(1)$
2. $(\forall n f(n) \le g(n)) \implies f = O(g)$
3. $f = O(g) \implies a \cdot f = O(g)$
4. $f = O(g)$ and $g = O(h) \implies f = O(h)$
5. $f = O(h)$ and $g = O(h) \implies f + g = O(h)$
6. $f_1 = O(g_1)$ and $f_2 = O(g_2) \implies f_1 \cdot f_2 = O(g_1 \cdot g_2)$

Consequence:

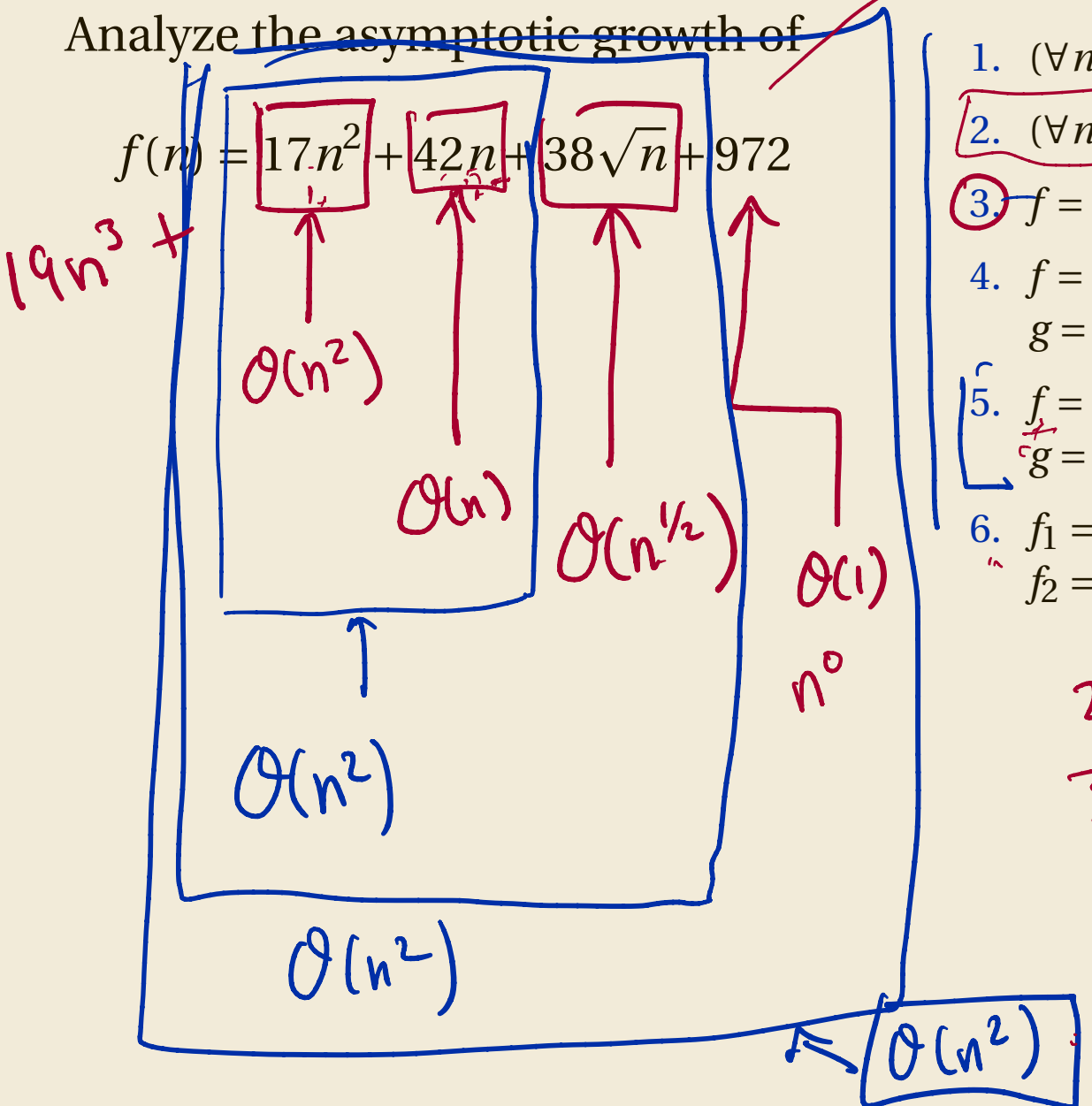
- If $a \le b$ then $n^a = O(n^b)$

Exercise. Show that if $a > b$, then $n^a \neq O(n^b)$.

Example with Functions

Analyze the asymptotic growth of

$$f(n) = 17n^2 + 42n + 38\sqrt{n} + 972$$



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$O(n^{100})$

$2 \Rightarrow n^2 = O(n^2)$

$3 \Rightarrow$ wl $a = 17$

$17n^2 = O(n^2)$

Example with Pseudocode

Example. Analyze the *worst-case* running time of the INSERTIONSORT procedure defined below.

- *assume* elementary operations have running time $O(1)$

```
1: procedure InsertionSort(a, n)
2:     for i = 2, 3, ... n do
3:         j ← i
4:         while j > 1 and a[j − 1] < a[j] do
5:             Swap(a, j − 1, j)
6:             j ← j − 1
7:         end while
8:     end for
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```

n = size of array

a = array

$O(1)$

$O(1)$

$O(1)$

$\leq i$ iterations

$\leq n$

$O(i) = c \cdot i$

Inner loop time $= O(n)$

Analysis of outer loop

$\leq n$ iterations

Overall running time is $O(n^2)$.

Best Case Running Time

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```

Consider sorted a

$a[1] \leq a[2] \leq a[3] \leq \ldots$

not satisfied if sorted

So $\Theta(1)$

PollEverywhere Question

What is the **best case** running time for InsertionSort? What arrays incur this running time?

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Variations of O

- $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$
  - Example: $4n^2 + 3n + 7 = \Theta(n^2)$
- $f = \Omega(g)$ if $g = O(f)$
  - Example: $0.01n^2 - 7n = \Omega(n^2)$
- $f = o(g)$ if for every $\varepsilon > 0$, there exists N_0 such that $n \geq N_0 \implies \frac{f(n)}{g(n)} < \varepsilon$.
  - Equivalently: $f = o(g) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
  - Example: $n^{1.999} = o(n^2)$
- $f = \omega(g)$ if $g = o(f)$
  - Example: $0.01n^{2.01} = \omega(n^2)$

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Mnemonic for Variations

Big-O	(in)equality
ω	$>$
Ω	$\geq$
Θ	$\approx$
O	$\leq$
o	$<$

Interpretation

Suppose:

- two algorithms A and B for solving the same problem
- running time of A is f, running time of B is g
- $f = o(g)$

Consider running A on a slow machine M_1 and B on a fast machine M_2. Then: regardless of how much slower M_1 is than M_2, for *sufficiently large* inputs, A will complete faster than B.

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- running time of A is f, running time of B is g
- $f = o(g)$

Consider running A on a slow machine M_1 and B on a fast machine M_2. Then: regardless of how much slower M_1 is than M_2, for *sufficiently large* inputs, A will complete faster than B.

The Moral. Efficient *algorithms* are better than faster hardware.

- little-o notation gives the "right" abstraction to formalize this relationship

Common Orders of Growth

Named orders of growth:

name	asymptotic growth
constant	$O(1)$
logarithmic	$O(\log n)$
polylogarithmic	$O(\log^c n)$
linear	$O(n)$
almost linear	$O(n \log^c n)$
quadratic	$O(n^2)$
polynomial	$O(n^c)$
exponential	$O(c^n)$

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Relationships

Between classes:

For all $a, b > 0$

- $a = o(\log^b n)$
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Relationships

Between classes:

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Within classes:

For all a, b, $a < b$

- $\log^a n = o(\log^b n)$
- $n^a = o(n^b)$
- $a^n = o(b^n)$

Next Time

- Abstract Data Types
- Fundamental Data Structures

Scratch Notes

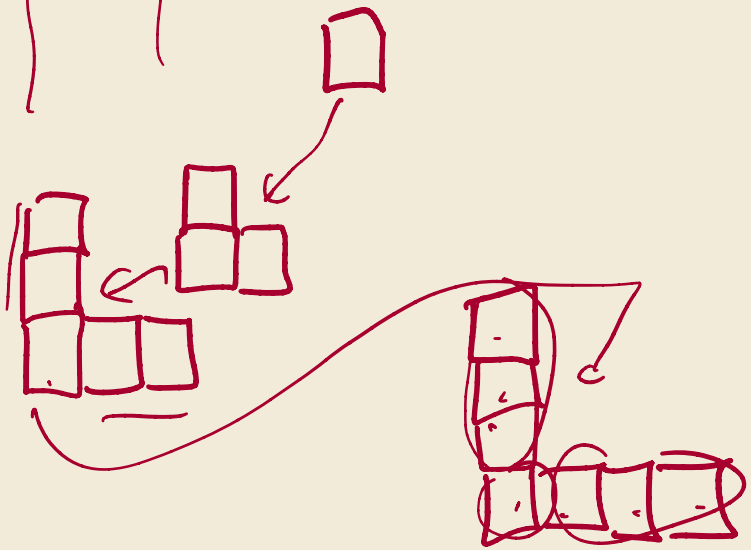
$M(5) = 2*5 - 1 +$

$M(4)$

$1+3+5+7 \cdots + 2k-1 = k^2$

$2\times4-1 + M(3)$

$2\times3-1 + M(2)$



Scratch Notes