Lecture 14: Computing the Mandelbrot Set

COSC 273: Parallel and Distributed Computing

Spring 2023

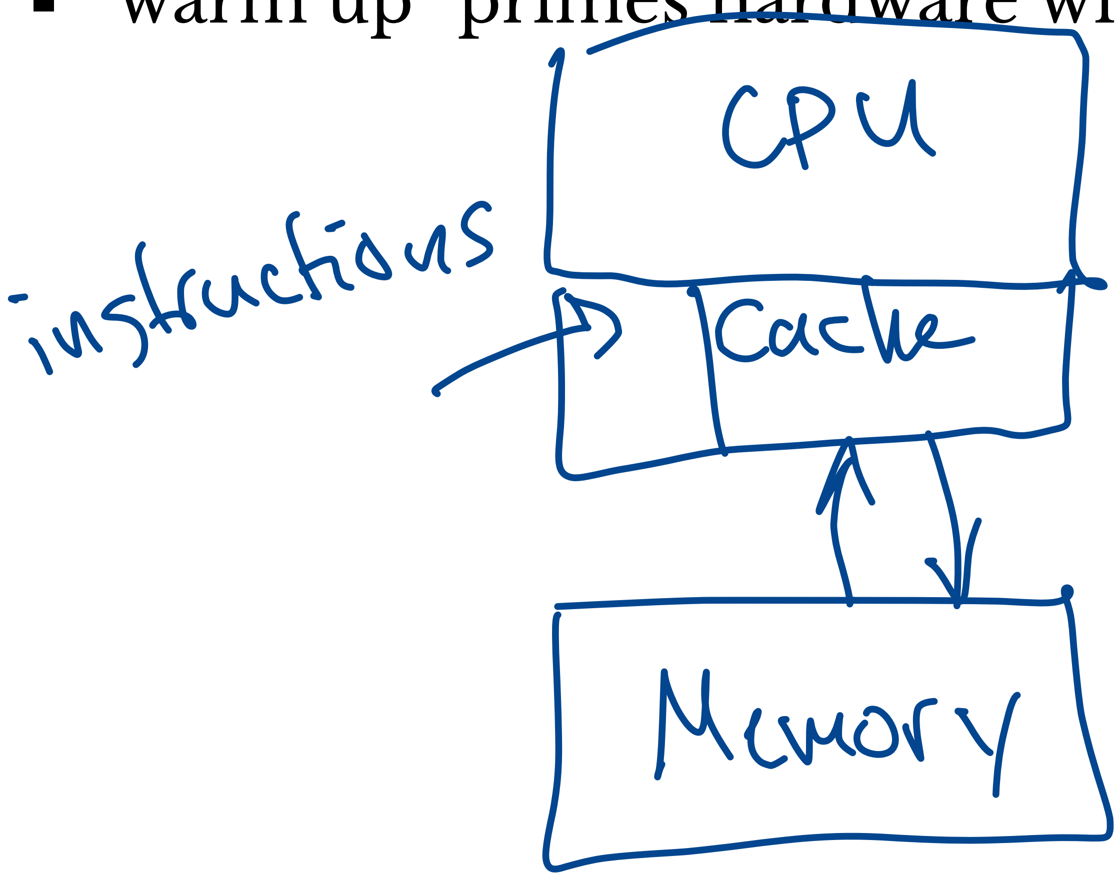
Outline

1. Benchmarking Notes
2. The Mandelbrot Set

Benchmarking Notes

To give "accurate" measure of efficiency:

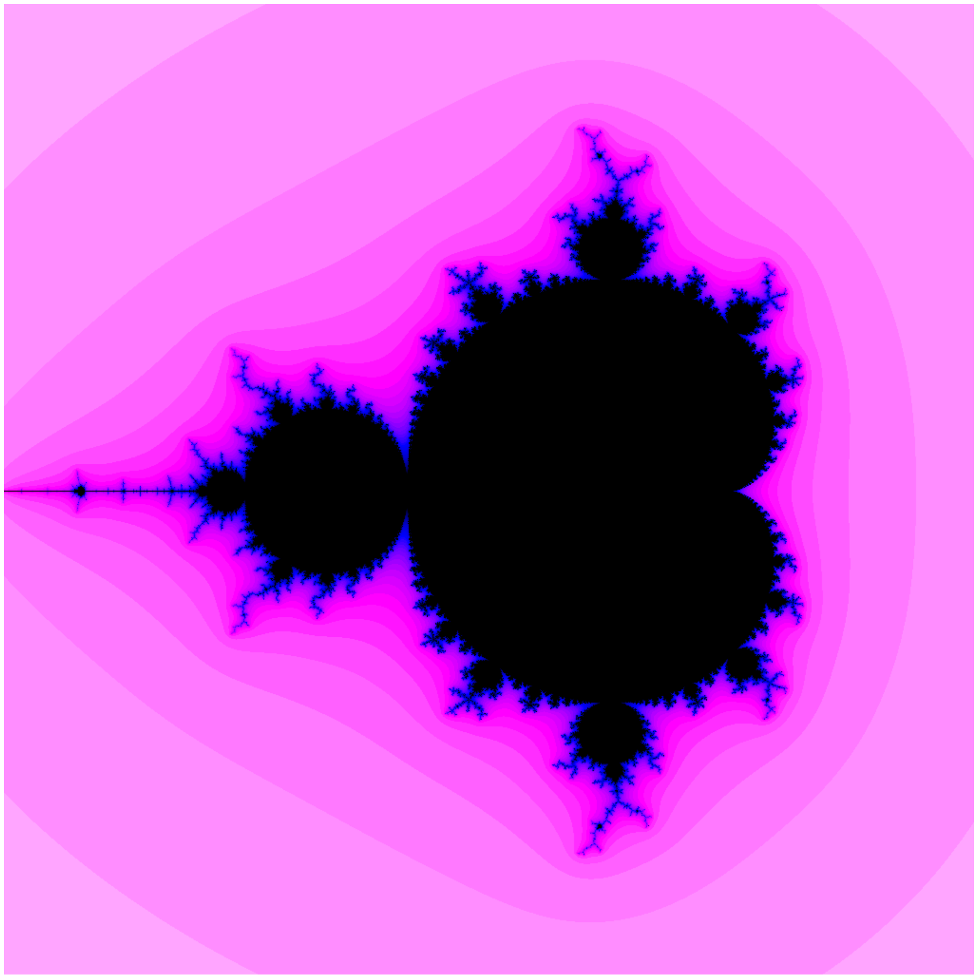
- test running time of method for **many** invocations
- run several invocations before starting timing
  - "warm up" primes hardware with correct instructions



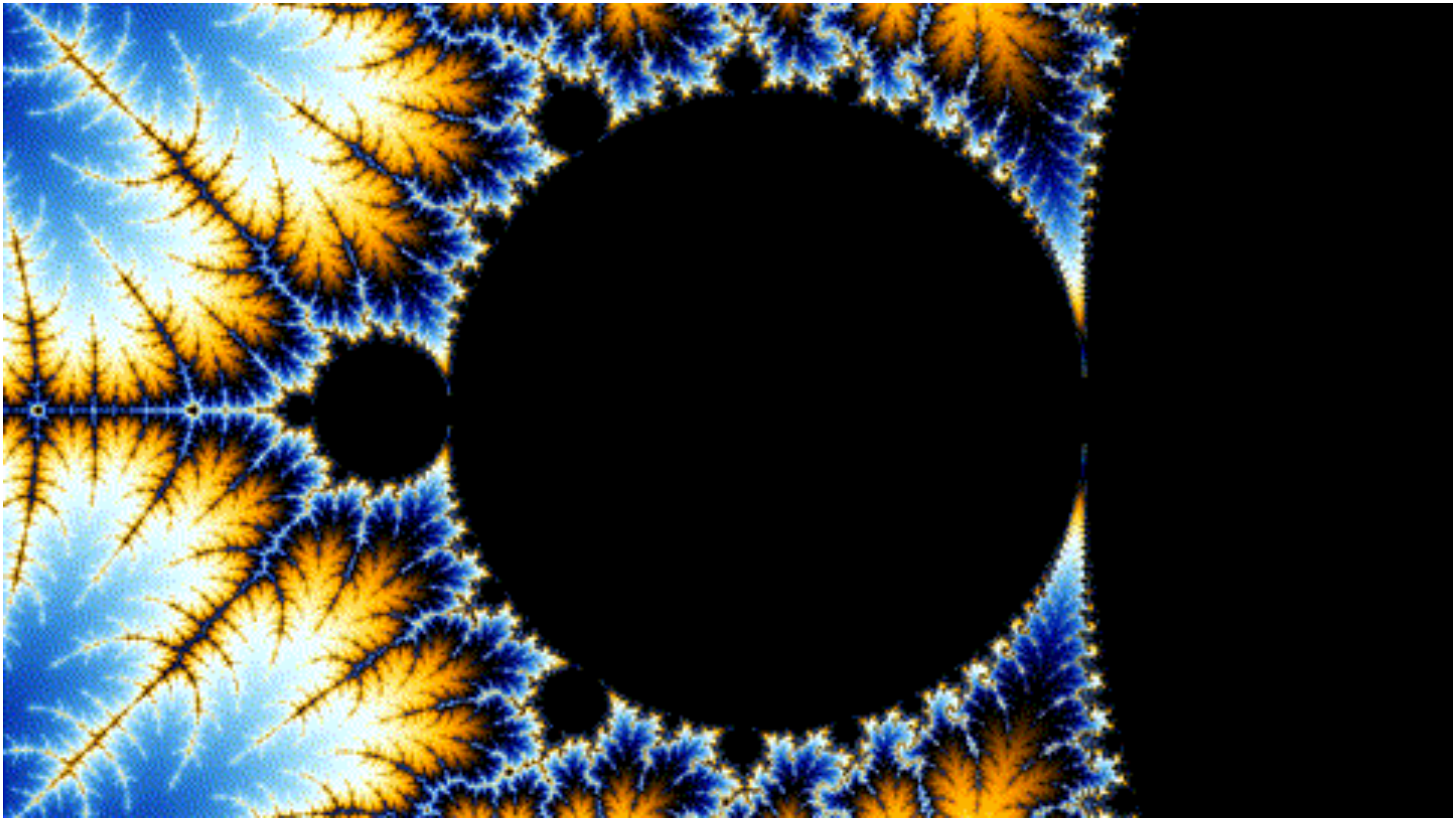
Benchmarking Demo

Lab 03: Mandelbrot Set

Draw this picture as quickly as possible!



Even Better: Animation



Source: Wikipedia

How?

The **Mandelbrot set** is a set of *complex numbers* that satisfy a certain property. We'll need:

1. (Re)view of complex numbers and arithmetic
2. Iterated maps and Mandelbrot set definition
3. Computing and visualizing the Mandelbrot set

Complex Numbers

$$(a+bi)(c+di) = ac + adi + bci + (bi)(di)$$
$$= (ac - bd) + (ad+bc)i$$

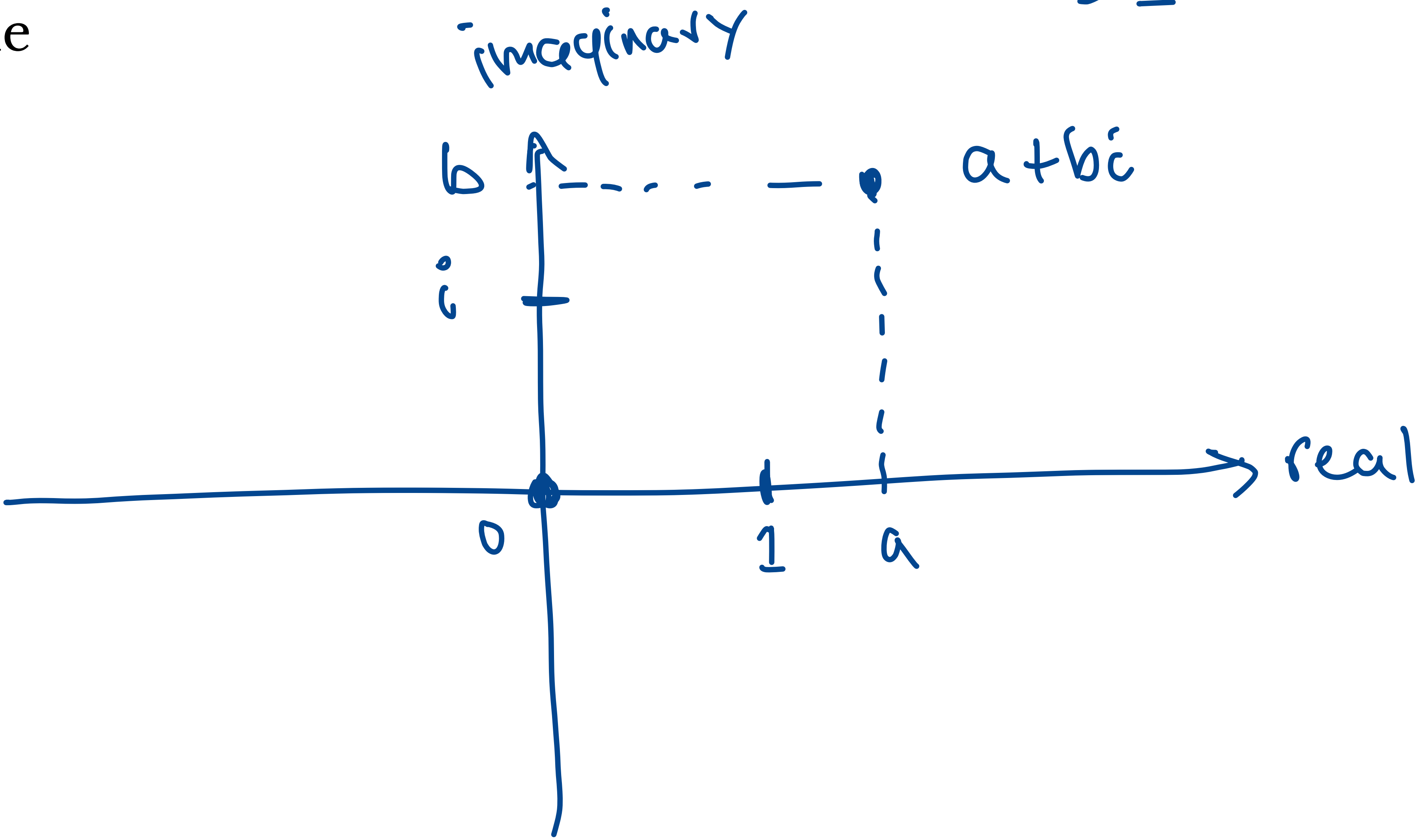
Complex Numbers

Recall:

- the imaginary number i satisfies $i^2 = -1$
- complex numbers are number of the form $a + bi$ where a and b are real
- complex arithmetic:
  - $(a + bi) + (c + di) = (a + c) + (b + d)i$
  - $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$
- modulus (or length):
  - $|a + bi| = \sqrt{a^2 + b^2}$

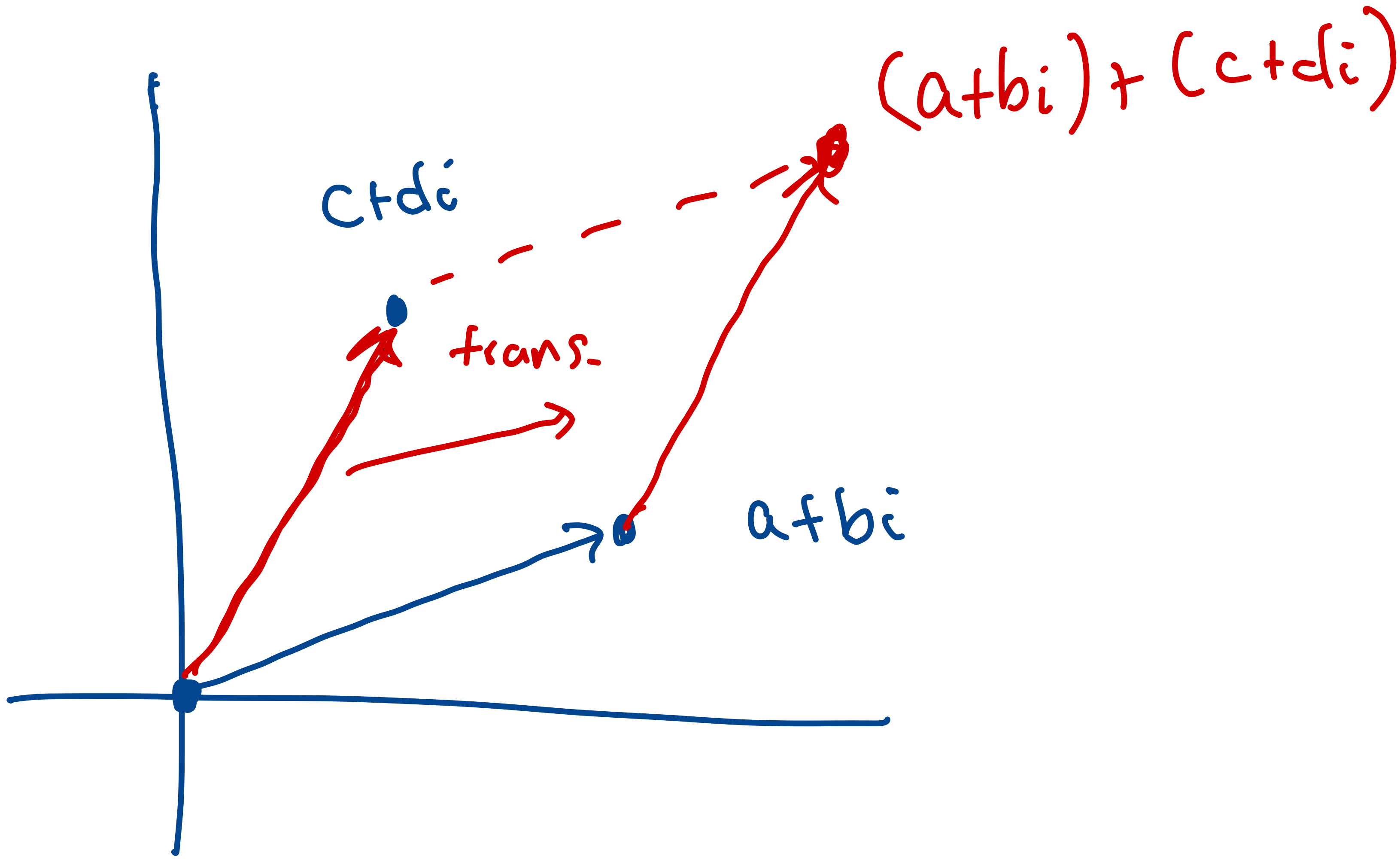
Complex Plane

Associate complex number $a + bi$ with point (a, b) in plane



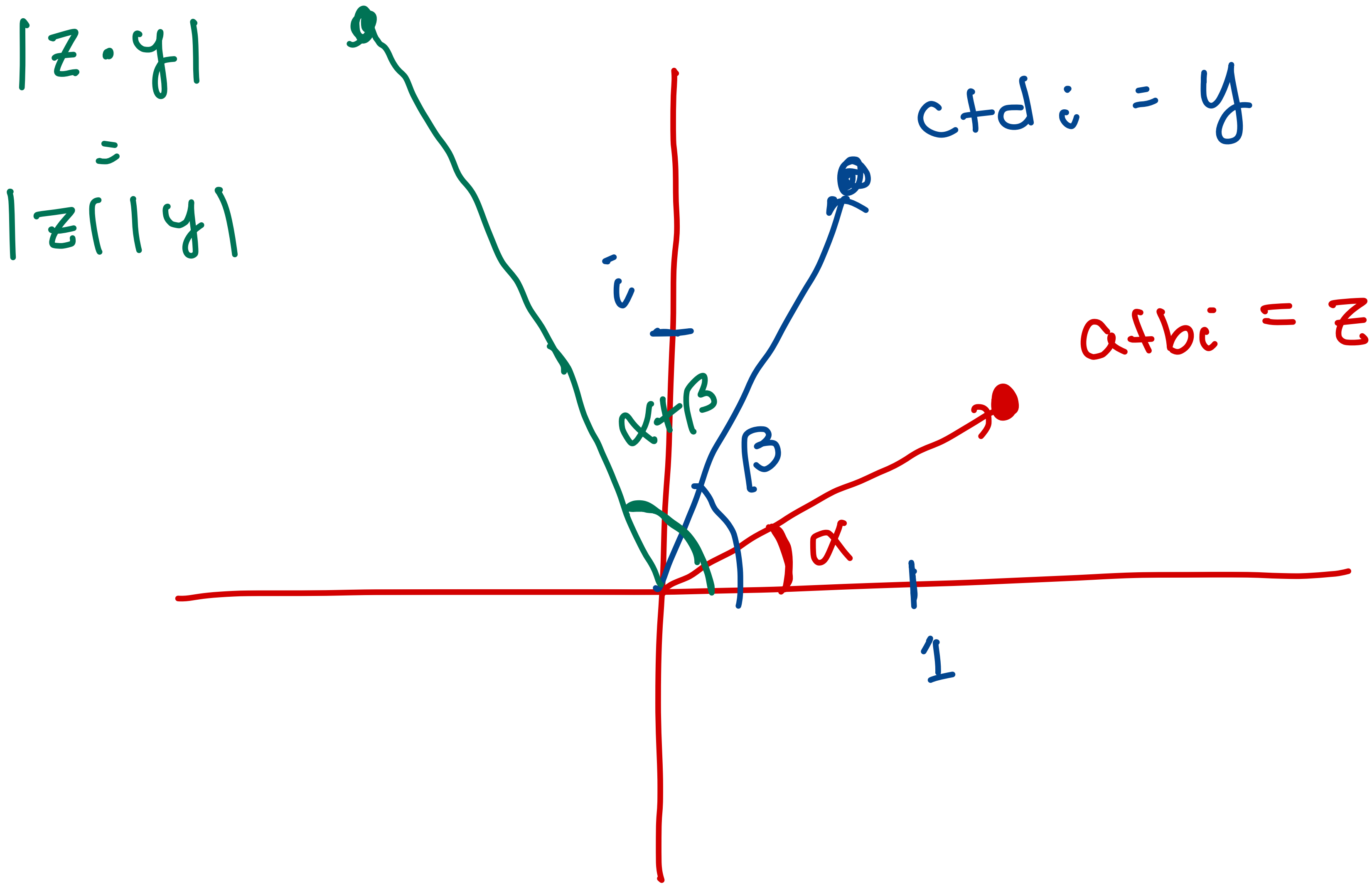
Geometric View of Addition

Addition is vector addition:



Geometric View of Multiplication

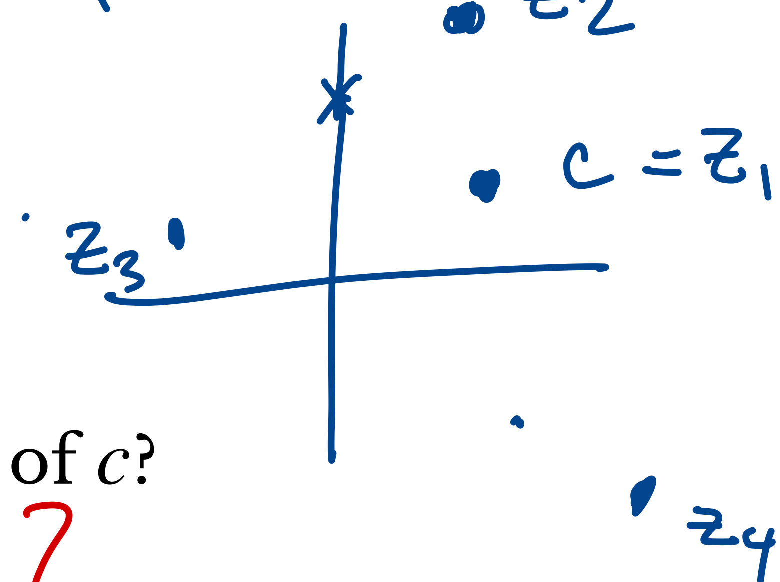
Multiplication adds angles, multiplies lengths:



Iterated Operations

Fix a complex number c starting value

- Define sequence $z_1, z_2, z_3, \ldots$ by
  - $z_1 = c$
  - for $n > 1$, $z_n = z_{n-1}^2 + c$
- What happens for different values of c?

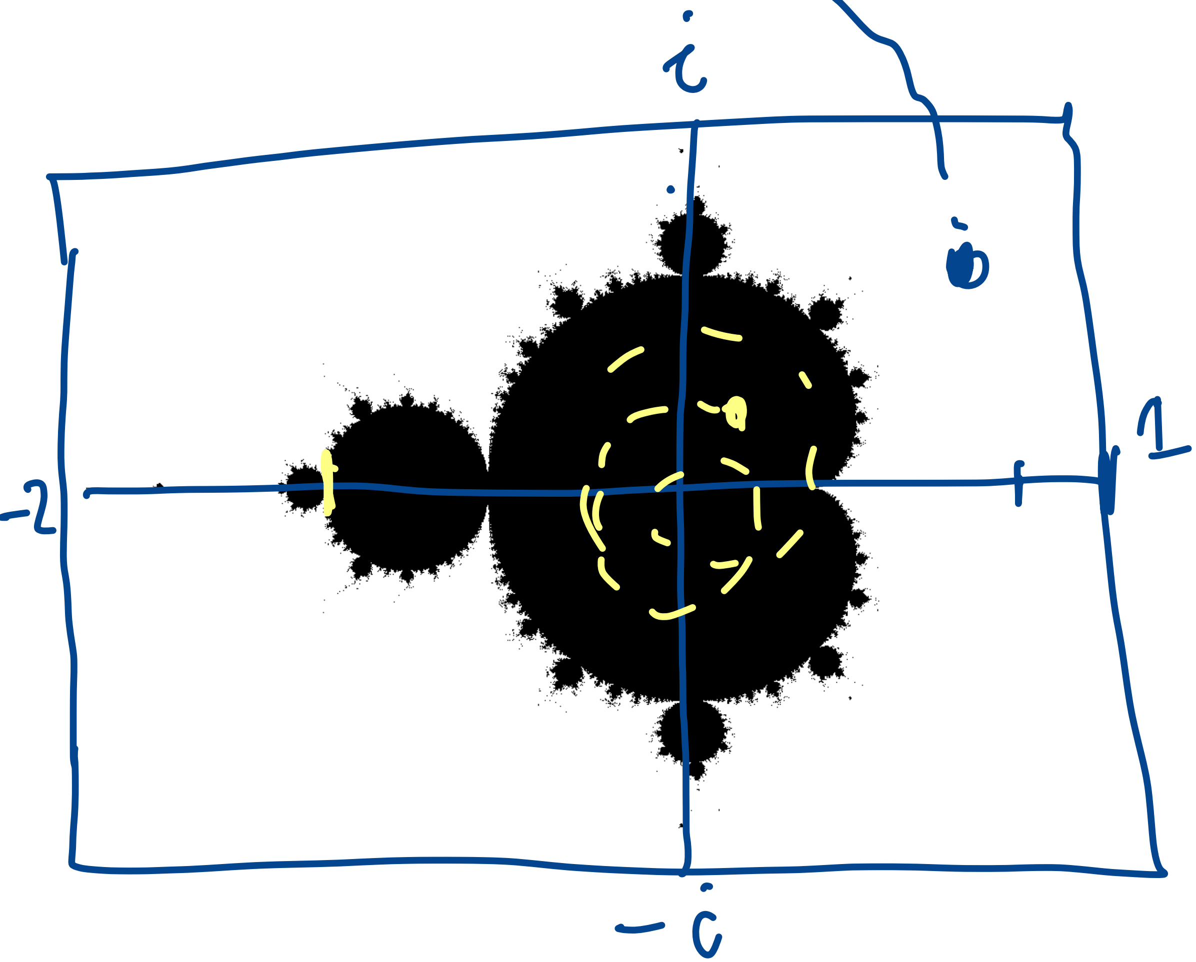


$c = 0$: $z_1 = 0$, $z_2 = 0$, $z_3 = 0$, $z_4 = 0$ — Bounded

$c = 1$: $z_1 = 1$, $z_2 = 2$, $z_3 = 5$, $z_4 = 25$ — Unbounded

Mandelbrot Set

The **Mandelbrot set** is the set M of complex numbers c such that the sequence $z_1, z_2, \ldots$ remains *bounded* (i.e., $|z_n|$ does not grow indefinitely)



Question

How can we determine if a given number $a + bi$ is in the Mandelbrot set?

yes

$c = -1$:

- $c = \frac{1}{2} = z_1$
- $z_2 = \left(\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{3}{4}$
- $z_3 = \left(\frac{3}{4}\right)^2 + \frac{1}{2} = \frac{17}{16}$
- $z_4 = \left(\frac{17}{16}\right)^2 + \frac{1}{2}$

$\vdots$ $\quad > 1$

$c = \frac{1}{2}$ is not in M.

$z_2 = 0$

$z_3 = -1$

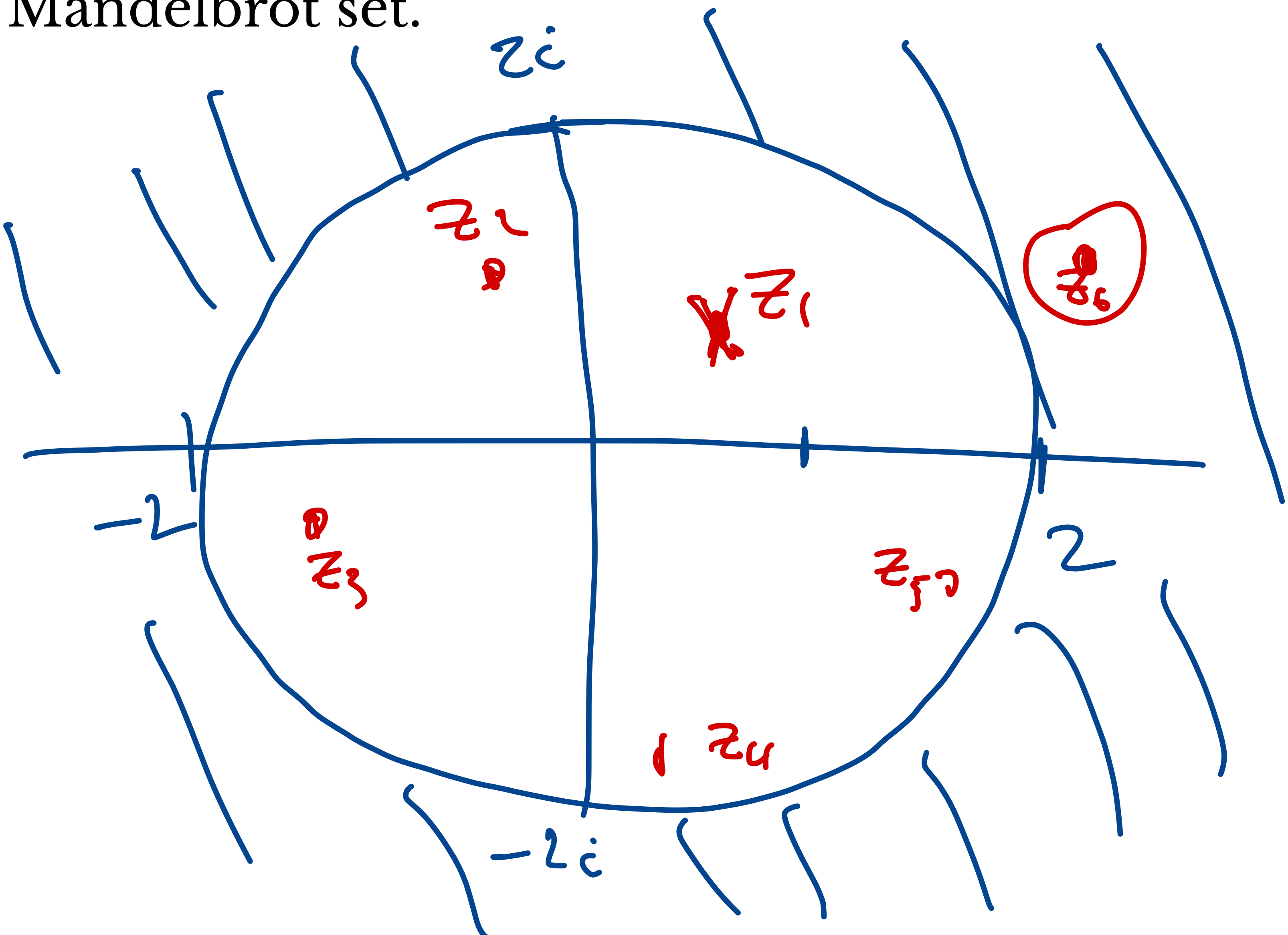
$z_4 = 0$

$z_5 = -1$

$z_6 = 0$

Facts

1. If c satisfies $|c| > 2$, then c is not in the Mandelbrot set.
2. If $|c| \leq 2$ but some z_n satisfies $|z_n| > 2$, then c is not in the Mandelbrot set.



Facts

1. If c satisfies $|c| > 2$, then c is not in the Mandelbrot set.
2. If $|c| \leq 2$ but some z_n satisfies $|z_n| > 2$, then c is not in the Mandelbrot set.

Observation. These facts give us a way of **excluding** points not in M. *How?*

compute $z_1, z_2, z_3, z_4, \ldots z_N$ (N = 100)

if any $|z_n| > 2$, then c is not in M

Activity

Determine if the following points are in the Mandelbrot set:

- -1
- i
- $i + 1$
- -2

Depicting the Mandelbrot Set

Make a grid of pixels!

Computing the Mandelbrot Set

Choose parameters:

- N number of iterations
- M maximum modulus ($M > 2$)

Given a complex number c:

- compute $z_1 = c, z_2 = z_1^2 + c, \ldots$ until
  1. $|z_n| \geq M$
     - stop because sequence appears unbounded
  2. Nth iteration
     - stop because sequence appears bounded
- if Nth iteration reached c is likely in Mandelbrot set

Illustration

https://complex-analysis.com/content/mandelbrot_set.html

Drawing the Mandelbrot Set

- Choose a region consisting of $a + bi$ with
  - $x_{min} \leq a \leq x_{max}$
  - $y_{min} \leq b \leq y_{max}$
- Make a grid in the region
- For each point in grid, determine if in Mandelbrot set
- Color accordingly

Counting Iterations

Given a complex number c:

- compute $z_1 = c, z_2 = z_1^2 + c, \ldots$ until
  1. $|z_n| \geq M$
     - stop because sequence appears unbounded
  2. Nth iteration
     - stop because sequence appears bounded
- if Nth iteration reached c is likely in Mandelbrot set

Color by Escape Time

1. Color black in case 2 (point is in Mandelbrot set)
2. Change color based on n in case 1:
   - smaller n are “farther” from Mandelbrot set
   - larger n are “closer”

Lab 03

Input:

- A square region of complex plane

Output:

- Escape times for a grid of points in the region
- A picture of corresponding region

Goal:

- Compute escape times as quickly as possible