Lecture 09: Convex Hulls

COSC 225: Algorithms and Visualization

Spring, 2023

Annoucements

1. Office hours canceled today
2. Re-submission for assignments 1 and 2 open Wednesday to Friday — up to 90%
3. Assignment 06 can be done in pairs!
   - partner questionnaire today by Wednesday
   - Assignment 06 due 03/24

Outline

1. Convex Hulls
2. Activity: Finding the Convex Hull
3. Graham's Scan Algorithm

Last Time

Depth-first Search: A Case Study in Visualizing Algorithms

- Building a graph interactively
- Stepping through an execution
- Visualizing each step

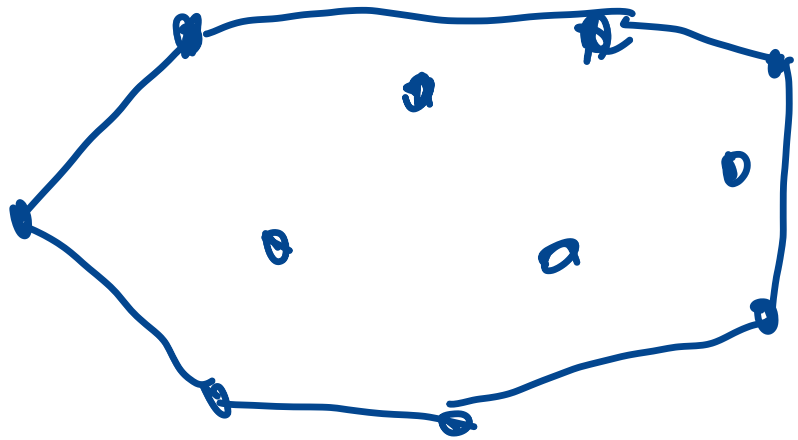
Today

Convex hulls:

- fundamental problem in computational geometry
- topic of Assignment 06

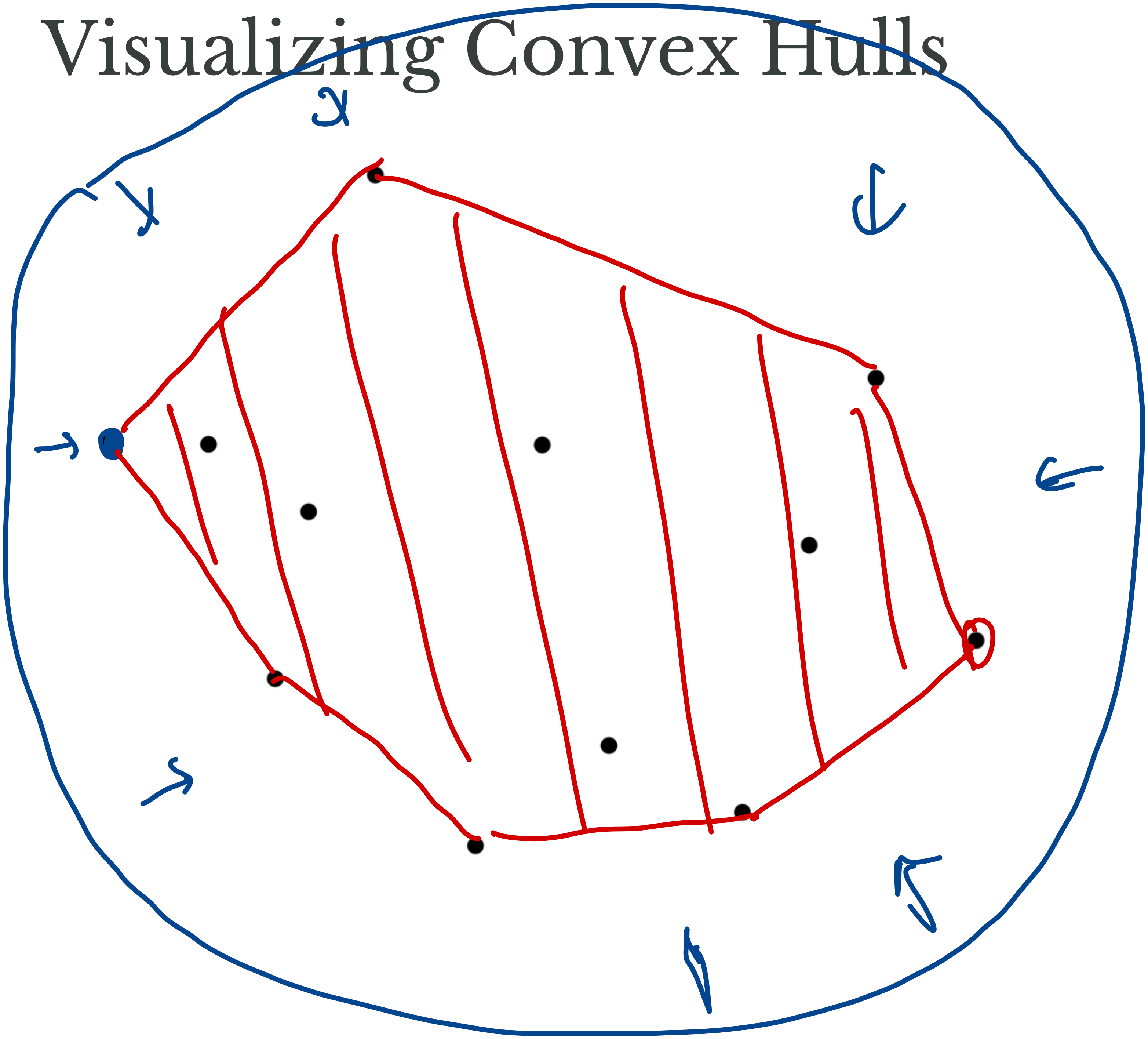
Convex Hulls, Two Views

Given a finite set of points in the plane:

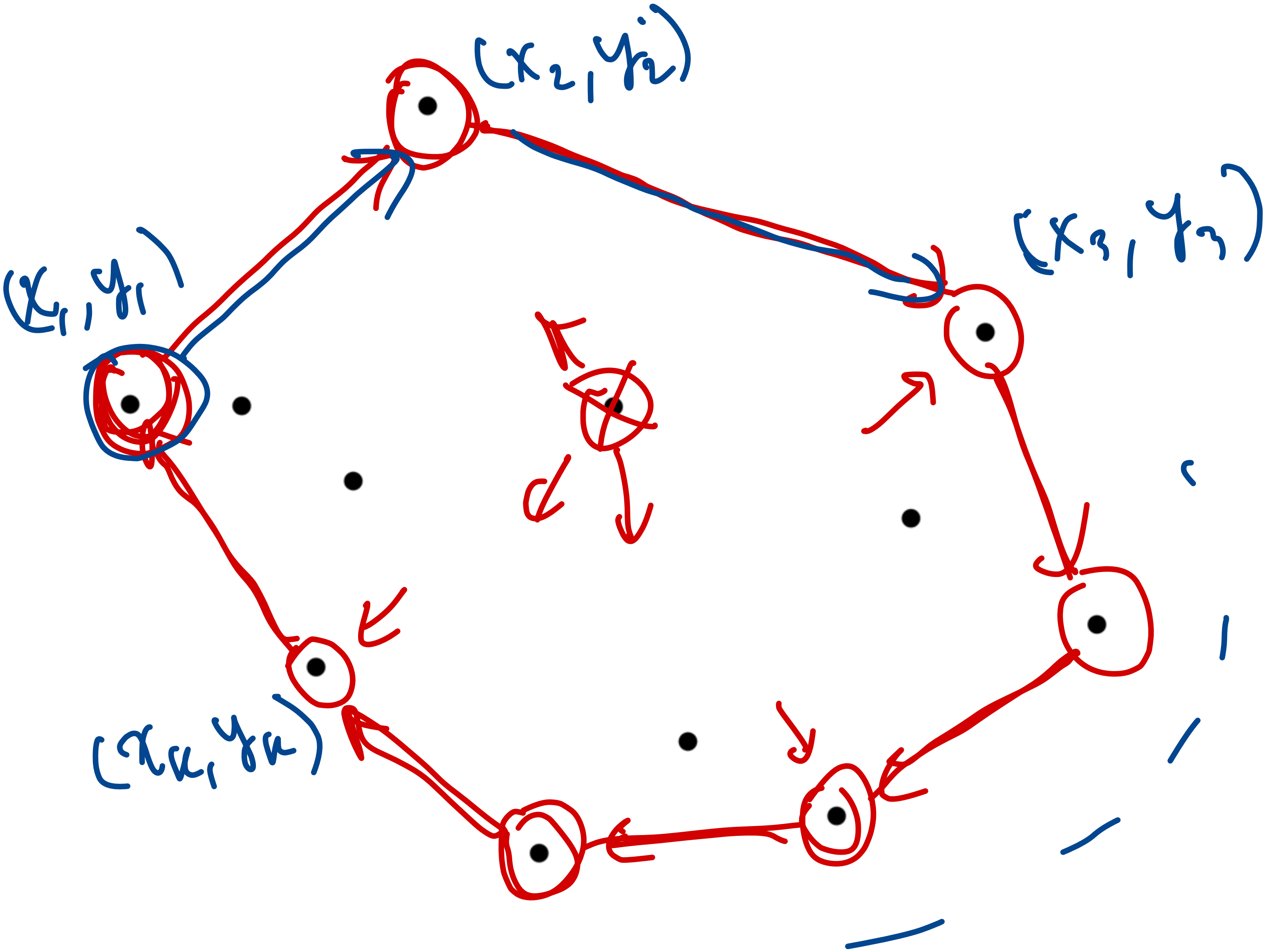


- What is the smallest **convex** region that contains all of the points?
- What is the minimum perimeter of a polygon that contains the points?

Visualizing Convex Hulls



Which Points are on the Boundary?



Convex Hull Problem

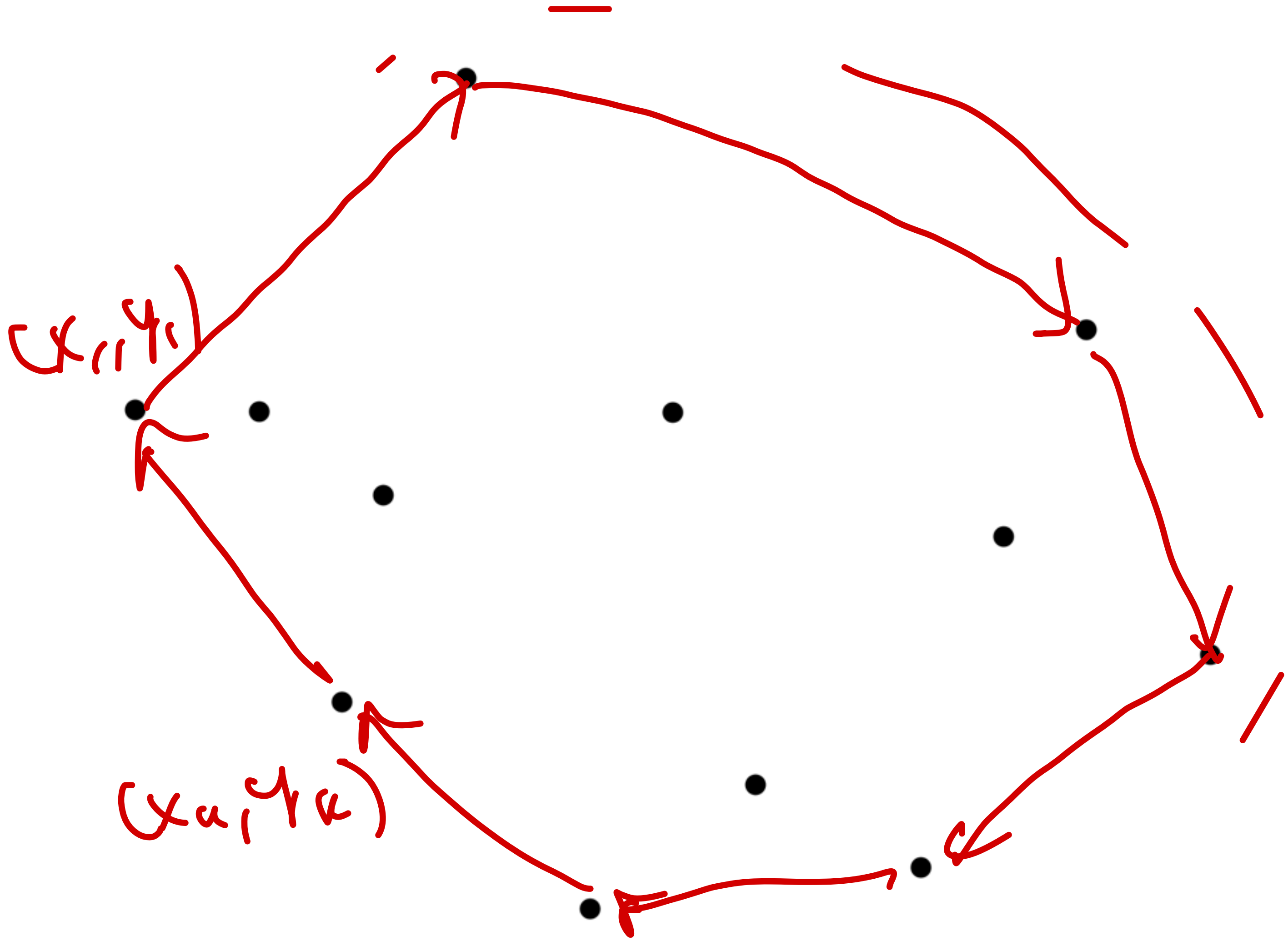
Input:

- set of points in plane
  - (x, y)-coordinates of each point

Output:

- a sequence of points $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ that define the "boundary" of the set of points
  - path around $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ surrounds all points in the set in clockwise order
  - the bounded region is **convex**

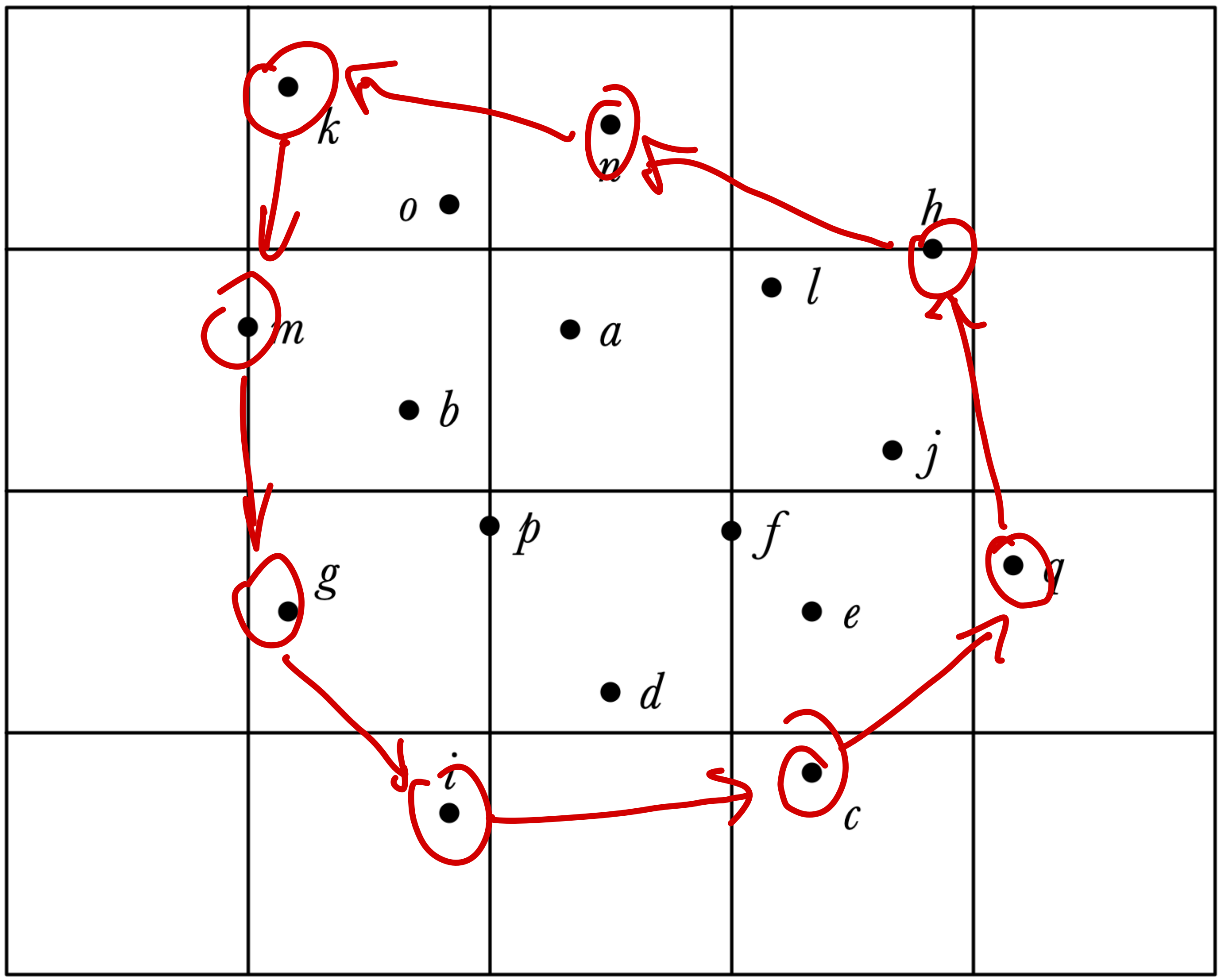
Convex Hull Output Depiction



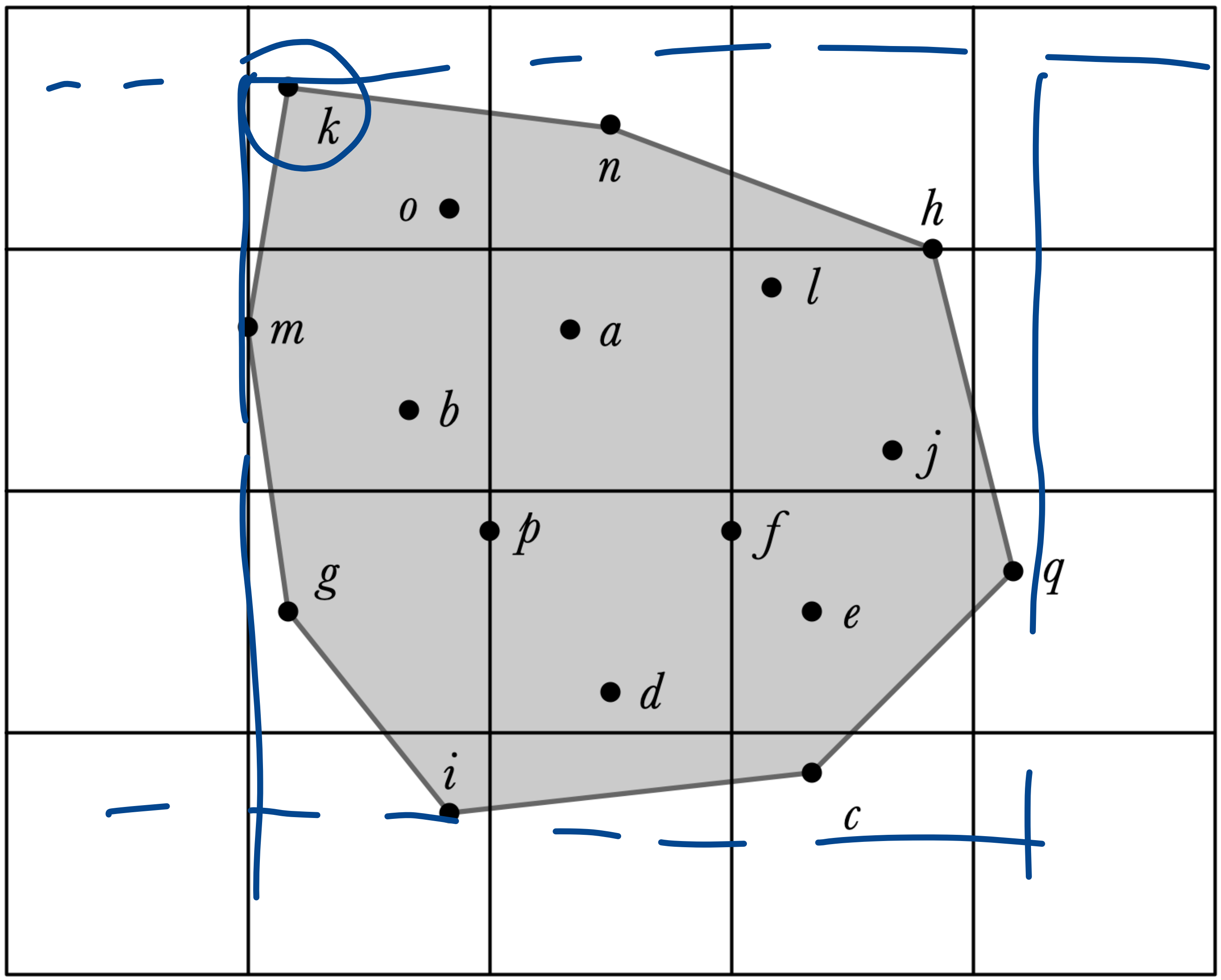
Activity!

Find the Convex Hull!

Graphed Points



With the Convex Hull



Observe

It is easy to find by hand once you've graphed the points.

Question. How to program a computer to find the convex hull?

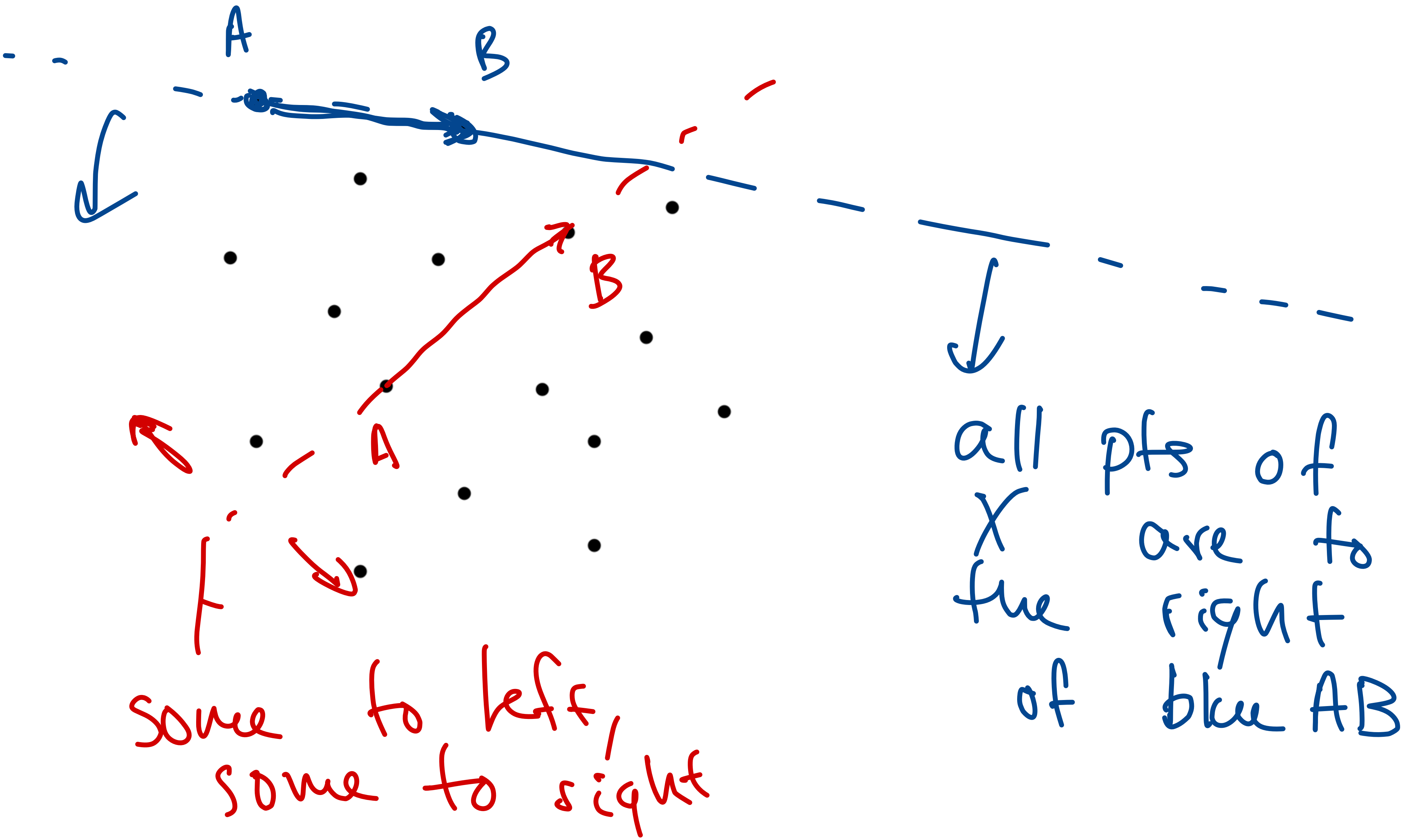
- using min/max
- iteratively add
- or start w/all pts & remove
- central point

Notation

- $X = (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ a set of points
- $CH(X)$ is the convex hull of X
  - more specifically, sequence of points in X that are vertices of the convex hull in clockwise order
  - start from left-most boundary point

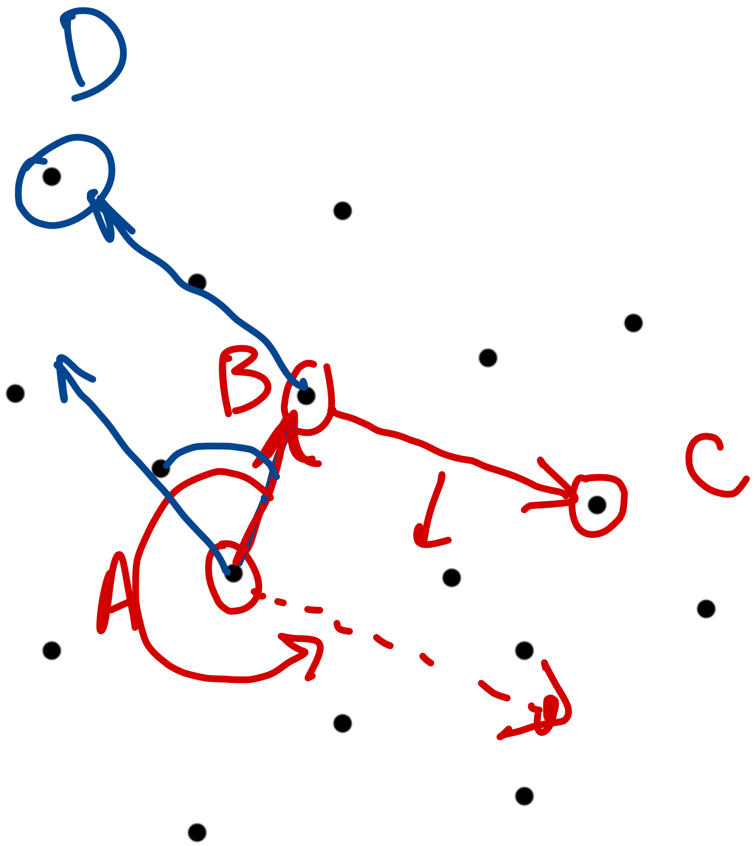
Initial Question

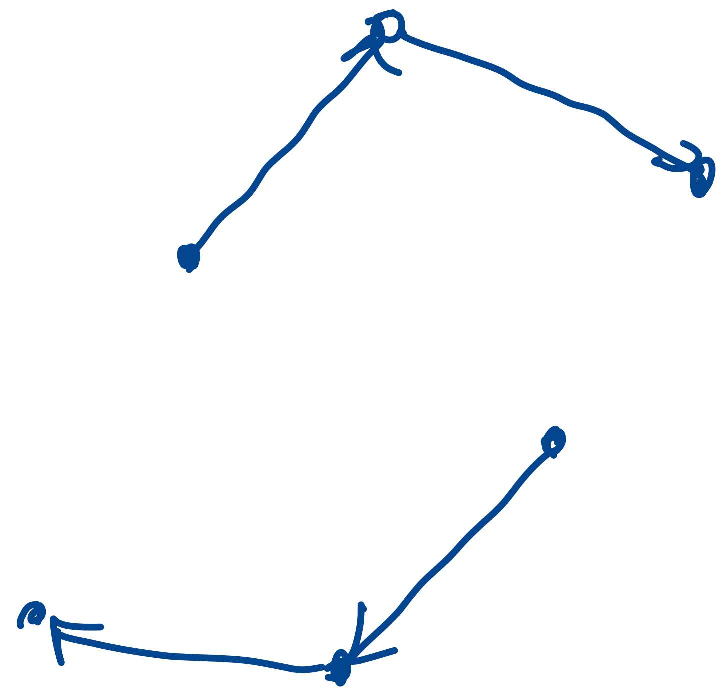
Given two points $A = (x_i, y_i)$, $B = (x_j, y_j)$ in X, how can we determine if AB is a (directed) segment of $CH(X)$?



Sub-Question

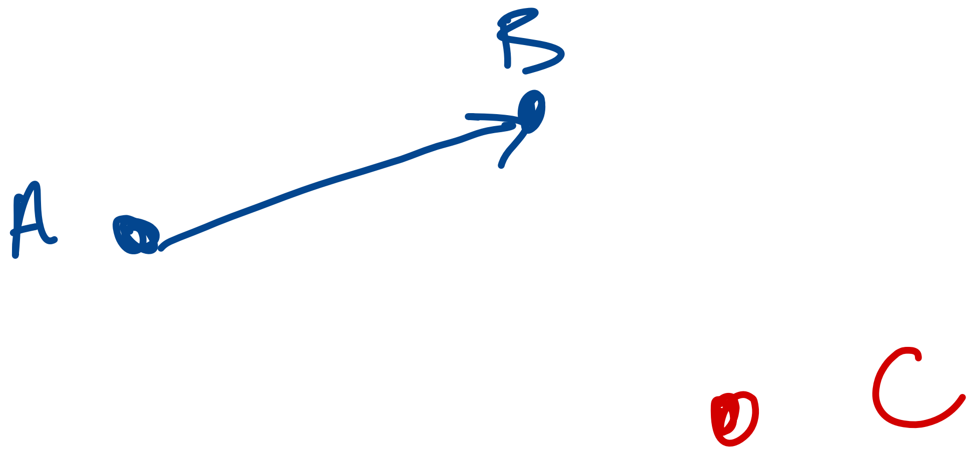
Given three points, A, B, C, how to determine if C lies "to the right" of AB?





Brute-force Solution

Check all ordered pairs of points AB to see if AB is a segment of the convex hull!



Check all points C and "accept" AB if all C are to the right of AB

Brute-force Solution

Check all ordered pairs of points AB to see if AB is a segment of the convex hull!

For all A and all $B \neq A$:

- check if each point $C \neq A, B$ is "to the right" of AB

assume $O(1)$ time

If all C are to the right of AB, add AB to the convex hull

Efficiency?

Question. If there are n points in X, what is the running time of the brute-force procedure?

- n choices for A
  - $n-1$ choices for B
    - $n-2$ choices for C

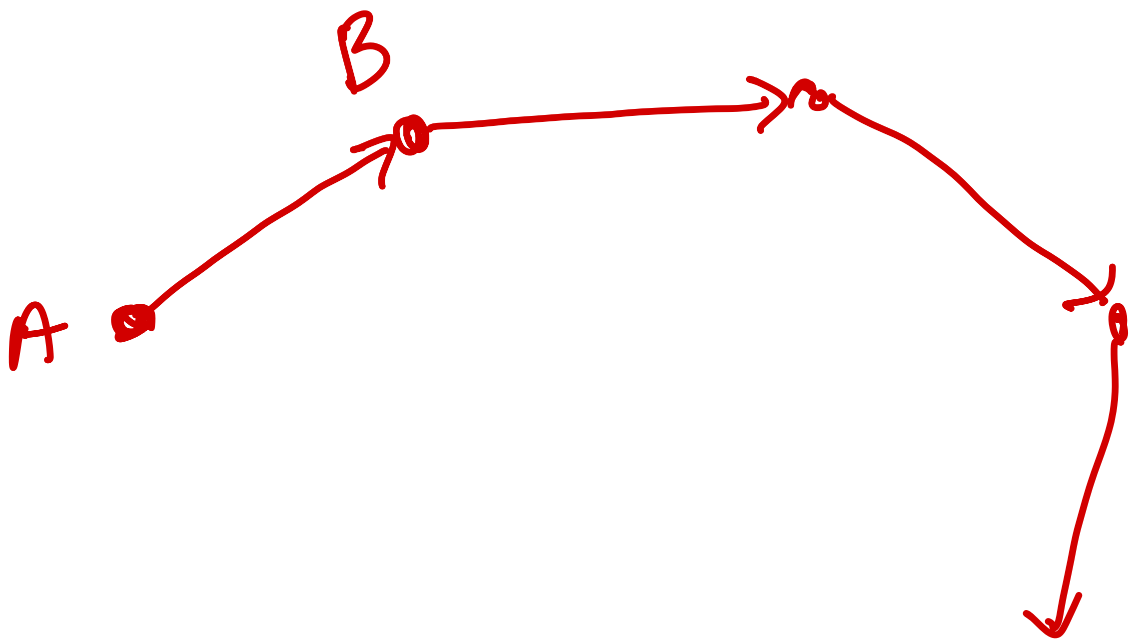
$\implies \Theta(n^3)$

Slight Optimization

Question. How could we ensure that we only ever check A that are **guaranteed** to be on the convex hull?

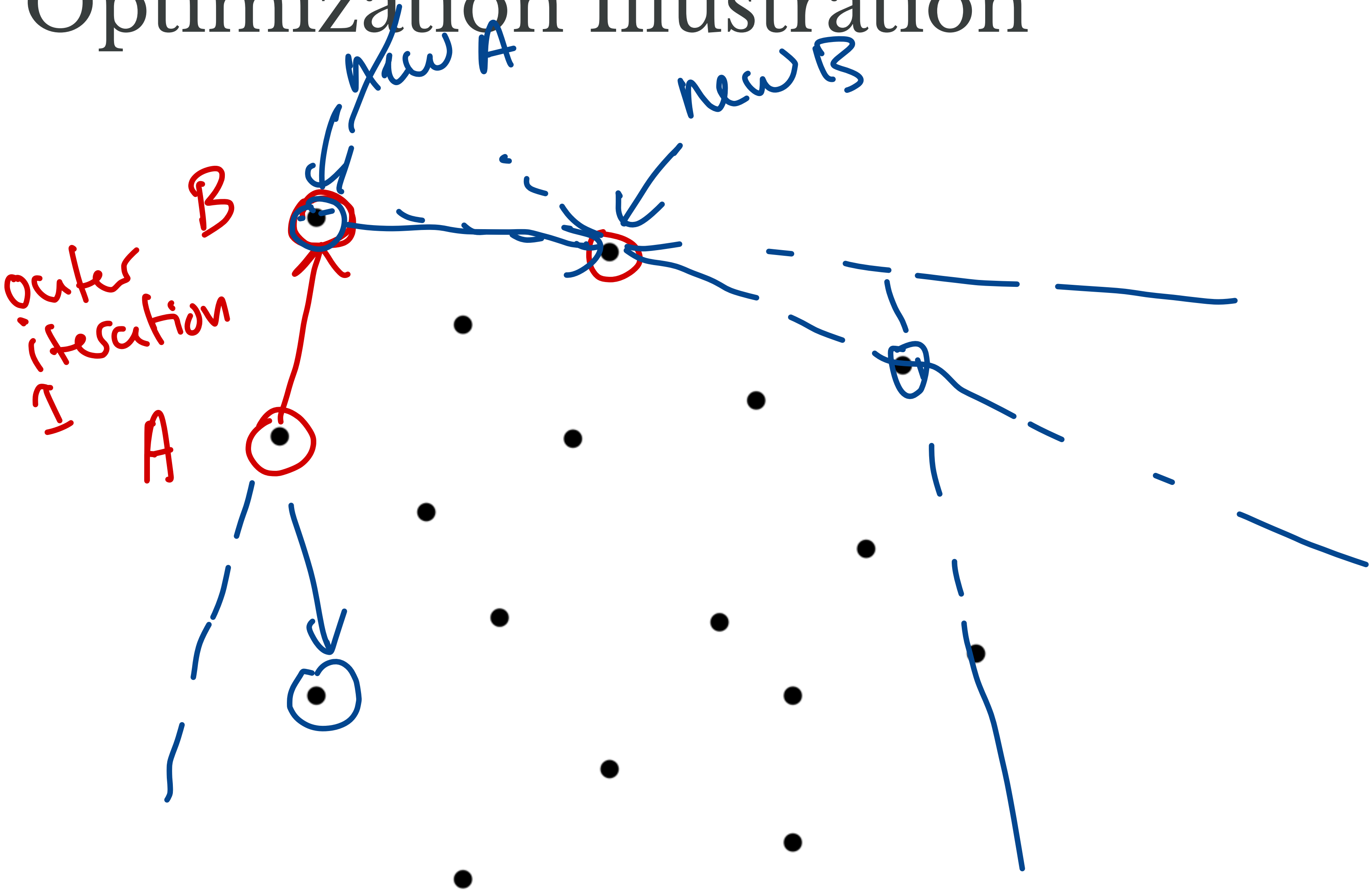
- Hint: how can we find our first A that is sure to be in $CH(X)$?

Start w/ A as leftmost point



for A
for B
for C

Optimization Illustration



Unfortunately

Even with the optimization, our code still takes $\Theta(n^3)$ time.

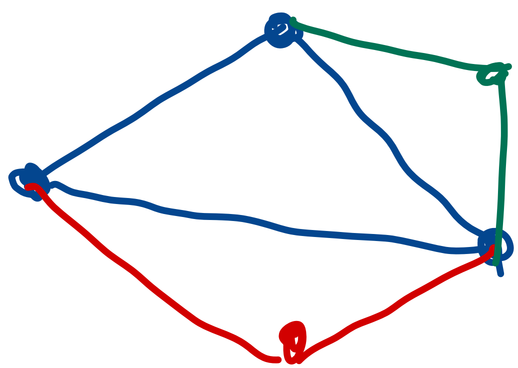
Bad example:

X is own convex hull

Another Approach

Try an **incremental algorithm**:

1. Start with a set X containing only one element
    - in this case finding $CH(X)$ is easy!
2. Add points one at a time to X and update $CH(X)$ accordingly



Another Approach

Try an **incremental algorithm**:

1. Start with a set X containing only one element
   - in this case finding $CH(X)$ is easy!
2. Add points one at a time to X and update $CH(X)$ accordingly

Questions.

1. In what order should we add points?
2. How do we update $CH(X)$ in response to point additions?

Graham’s Scan Algorithm

R. Graham, 1972

First Idea

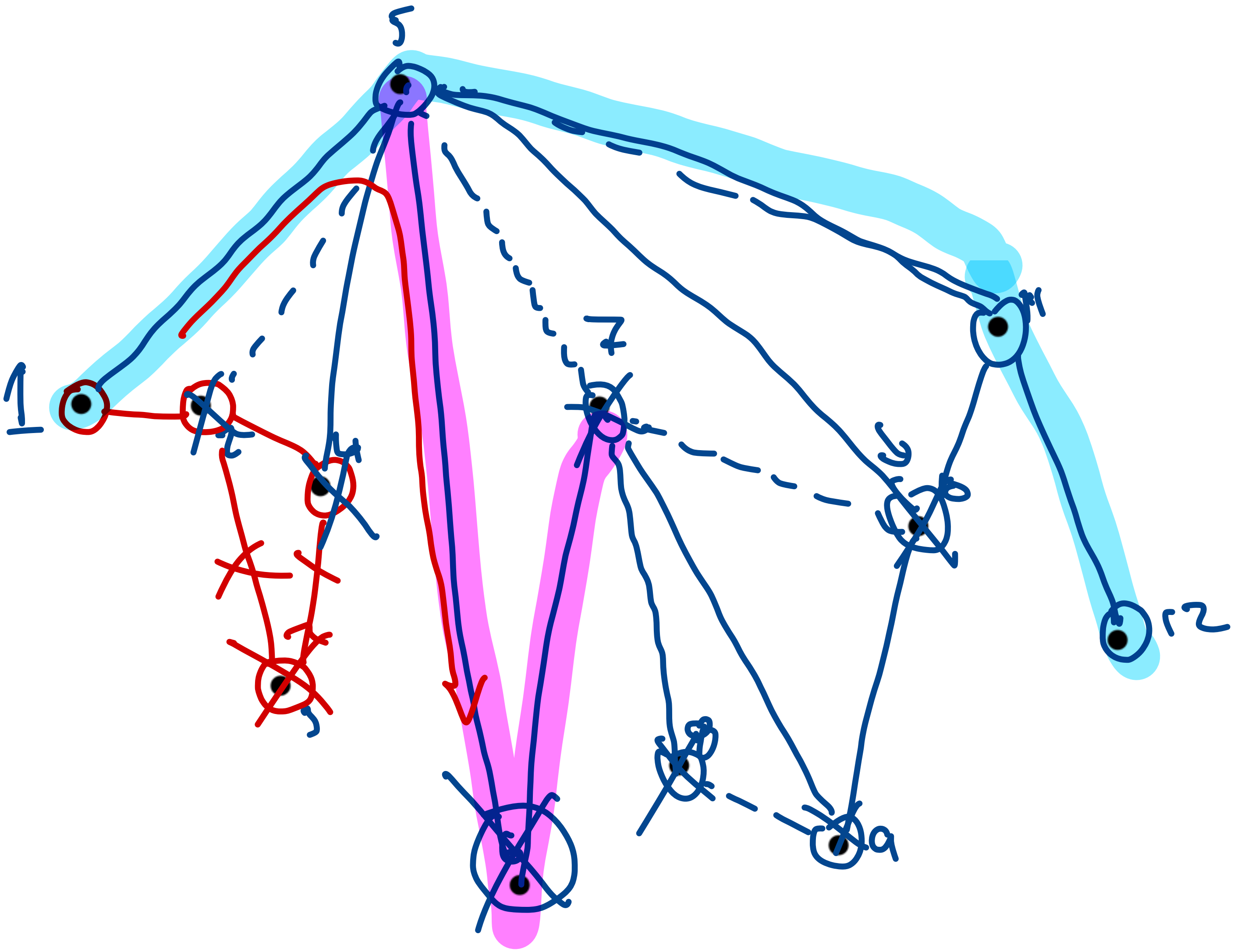
Pre-process points by sorting them **by x-coordinate**:

- process the points from left to right

Starting at left-most point, scan to the right to find **upper** convex hull boundary

- repeat process from right to left to get **lower** convex hull

Graham Scan Idea, Illustrated



How to Formalize/Implement?

Assume X is an array of points, sorted by x-coordinate

- How to keep track of points in $CH(X)$ (so far)?

  cur CH(X) is a stack

  ↳ "undo"

- How to update in response to next point?

  → if new point is "to right" push to stack

  → otherwise pop until new point is to right of to ~~two~~ elts on stack

Graham's Scan Pseudocode

- `X` sorted by x-coordinate
- `stk` a stack, initially storing first two points in `X`

For each remaining `C` in `X`:

- if `stk.size() == 1`, `stk.push(C)`
- otherwise
  - `A` and `B` are top two elements in `stk`
  - while `ABC` is not a right turn and `stk.size() > 1`
    - `stk.pop()`, update `A`, `B`
  - `stk.push(C)`

Claim

When Graham's Scan completes, `stk` stores the points along the upper boundary of the convex hull of X.

Why?

Claim

When Graham's Scan completes, `stk` stores the points along the upper boundary of the convex hull of X.

Why?

Must show:

1. Sequence of points in `stk` make only right turns.
2. All points in X are below path formed by points in `stk`

Graham's Scan Efficiency?

If there are n points, what is the running time of Graham's scan?

Finishing the Computation

How to find the **lower** boundary of $CH(X)$?

Assignment 06

Make an interactive visualization for Graham’s scan algorithm

- user can add points in the plane
- program steps through execution and illustrates each step
- returns convex hull of points