Lecture 05: Rounds, Neighborhoods + Lower Bounds

2sday
2/22/22

Remarks

- Office hours In C216 or C109
  - Weds. 10 – 11
  - T/Th. 11:20 – 11:50 *
  - by appointment

- Group work philosophy

- No lecture 04 recording (sorry!)

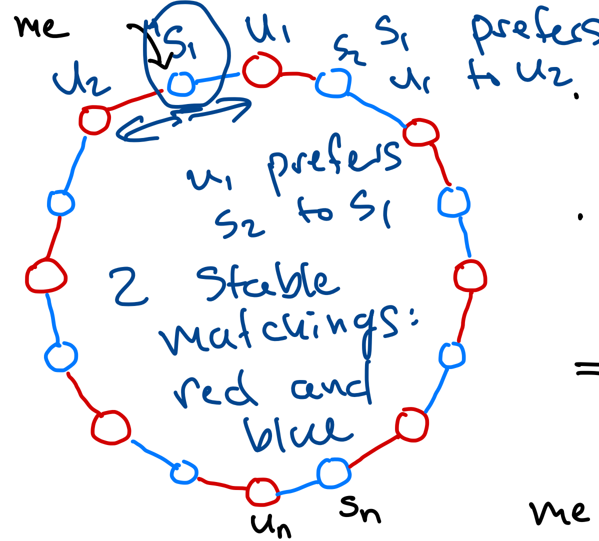
- Challenge Q's

Overview

1. Review of last time
    - troublesome SM instances
    - neighborhoods & protocols
2. Lower bound for SM
3. Proof of Locality Lemma
4. Maximal matchings

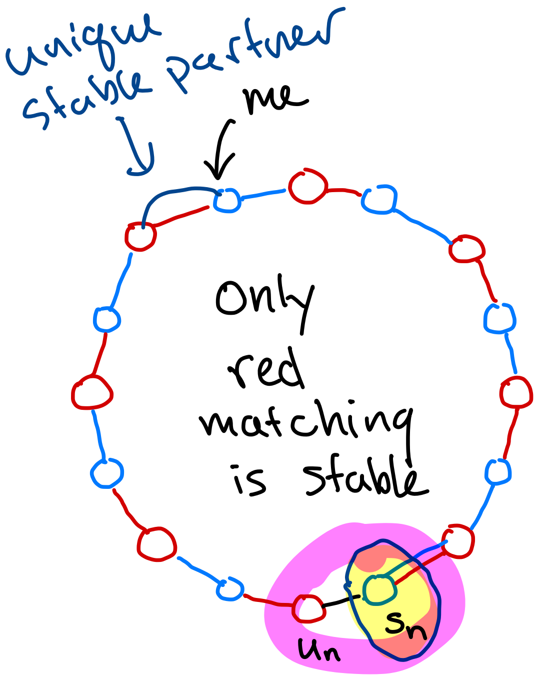
Last Time

- SM instances w/ n students, n internships
- each agent has 2 acceptable partners
- graph is a cycle



- all students prefer blue neighbor
- all internships prefer red neighbor

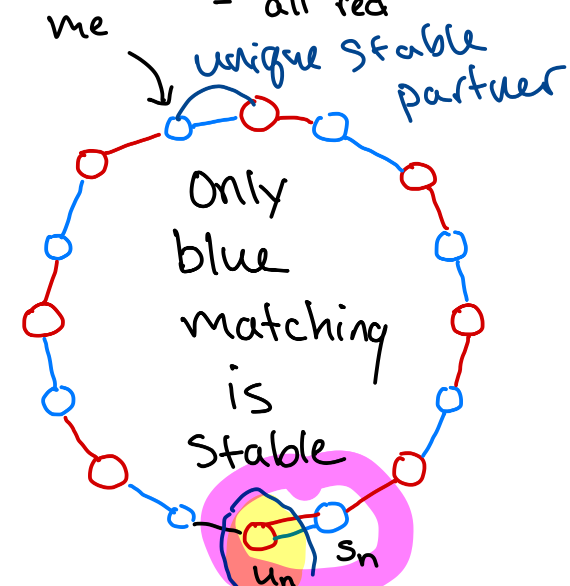
$\Longrightarrow$ 2 stable matchings:
- all blue
- all red



Changing only s_n's or u_n's prefs. force a <u>unique</u> SM

$\Downarrow$

my output (match) depends on s_n/u_n's input



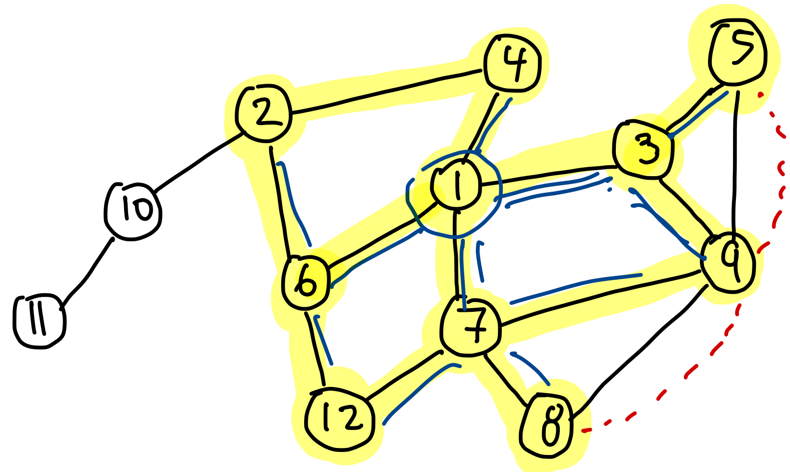
Also Last Time

1

$G = (V, E)$ a graph, $v \in V$ a vertex

$\Gamma_d(v)$ = v's distance d neighborhood

$$= \begin{cases} \text{all vertices within dist. } d \text{ from } v \\ \text{all edges w/ one endpt. within dist. } d-1 \end{cases}$$



$\Gamma_2(1)$

Today's Goal: Locality Lemma

- Fix graph $G = (V, E)$ and protocol Π
- for any $v \in V$ and round r, v's state in round r is determined by $\Gamma_{r-1}(v)$

{ Formalizes that information can't travel more than one "hop" per round }

But First: An application

Theorem. Consider SM instances w/ n students and n internships, and suppose Π is a protocol that computes a SM. Then Π requires at least $n-1$ rounds on some inputs.

cannot always find SMs in $< n-1$ rounds

Locality Lemma

- Fix graph $G = (V, E)$ and protocol Π
- for any $v \in V$ and round r, v's state in round r is determined by $\Gamma_{r-1}(v)$

Proof Idea

- argue by contradiction
  1. assume that some Π finds SM in $< n-1$ rounds on all inputs
  2. use this assumption to construct an input for which Π fails
- For 2, use indistinguishability argument
  - find 2 SM instances that Π cannot distinguish
    - use locality lemma here
  - but 2 instances require different (correct) outputs
  - therefore: Π fails for one of the inputs

Theorem. There are SM instances w/ n students and n internships such that any protocol Π requires at least $n-1$ rounds to compute a stable matching.

Proof.

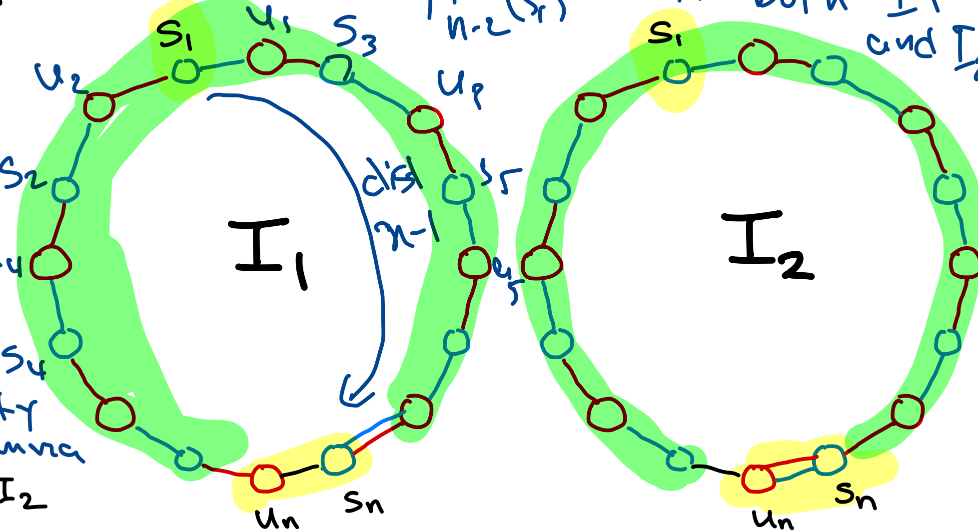
- Suppose Π a protocol terminating in $t(n) \leq n-1$ rounds
- consider instances I_1, I_2
  - same except prefs of u_n, s_n
  - $\mathrm{dist}(s_1, u_n), \mathrm{dist}(s_1, s_n) \geq n-1$
- consider output of Π at s_1
  - determined after $t(n) \leq n-1$ rnds
  - $\boxed{\text{determined by } \Gamma_{n-2}(s_1)}$ by Locality Lemma
- $\Gamma_{\cancel{n-1}\ n-2}(s_1)$ is same in I_1, I_2

  $\Rightarrow$ Π gives same output for I_1, I_2

Locality Lemma

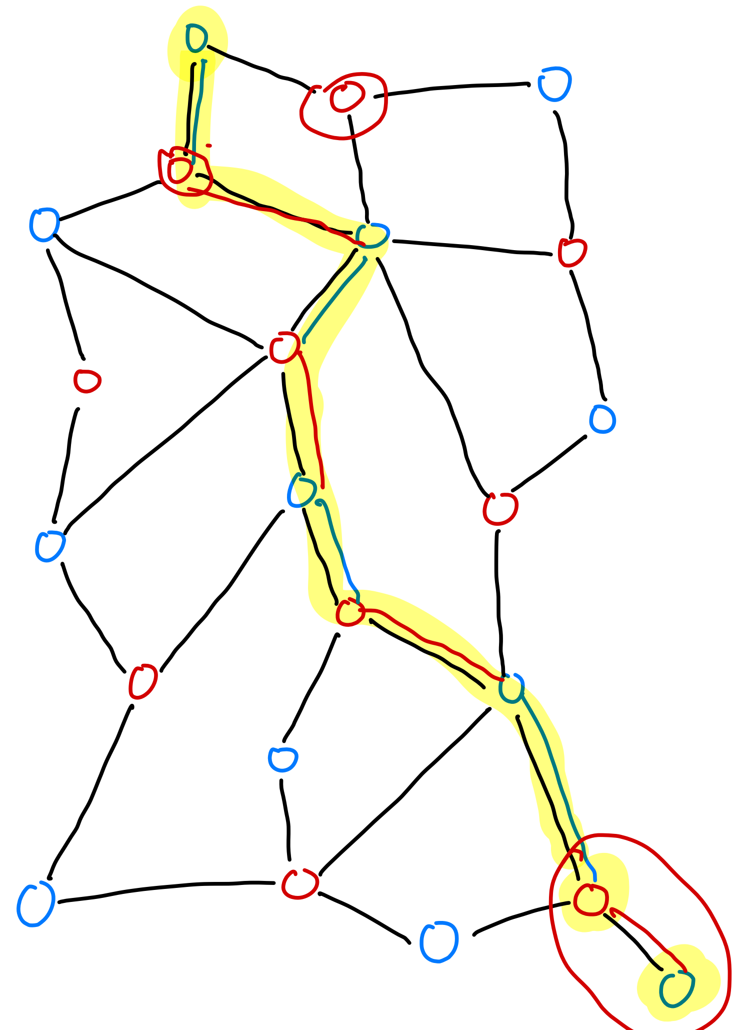
- Fix graph $G = (V, E)$ and protocol Π
- for any $v \in V$ and round r, v's state in round r is determined by $\Gamma_{r-1}(v)$

$\Gamma_{n-2}(s_1)$ = in both I_1 and I_2



- s_1's unique stable partners are different in I_1, I_2 $\Rightarrow$ Π fails for one

Theorem. There are SM instances w/ n students and n internships such that any protocol Π requires at least $n-1$ rounds to compute a stable matching.



Remarks on Proof.

1. Result holds even if every agent has at most 2 acceptable partners

2. Argument can be generalized to graphs other than cycles
   → on any graph w/ diameter D, finding an SM requires $D-1$ rounds

3. Argument shows SM cannot be solved locally — SM is a global problem

Proving the Locality Lemma

Idea. Connect graph structure (neighborhoods) to execution of a protocol.

Locality Lemma

- Fix graph $G = (V, E)$ and protocol Π
- for any $v \in V$ and round r, v's state in round r is determined by $\Gamma_{r-1}(v)$

Main ingredients:

- structure of neighborhoods
  - → v's dist. d neighborhood is union of v's neighbors dist. $d-1$ neighborhoods
    "neighborhood covering lemma"
- recall definition of protocol
  - → msgs determined by state; state determined from local input + received msgs
- argue by induction on round r

base case: lemma is true for $r = 1$ (from def. of protocol)

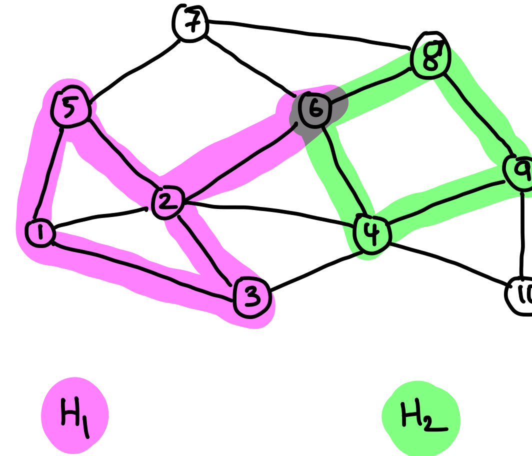
inductive step: If true for $r-1$, then also true in round r

Union of Subgraphs

Suppose H_1, H_2 are sub-graphs of a graph $G = (V, E)$. Then the union of H_1 and H_2, denoted $\boxed{H_1 \cup H_2}$, consists of

(1) all vertices in H_1 or H_2

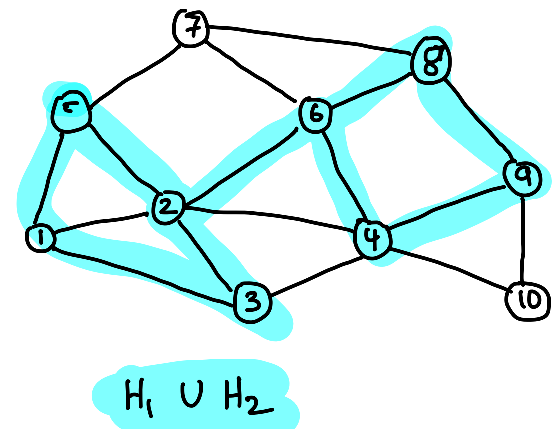
(2) all edges in H_1 or H_2



We can also form unions of many subgraphs:

$$\boxed{H_1 \cup H_2 \cup H_3 \cup \cdots \cup H_k = \bigcup_{i=1}^{k} H_i}$$

= vertices and edges in any of the subgraphs

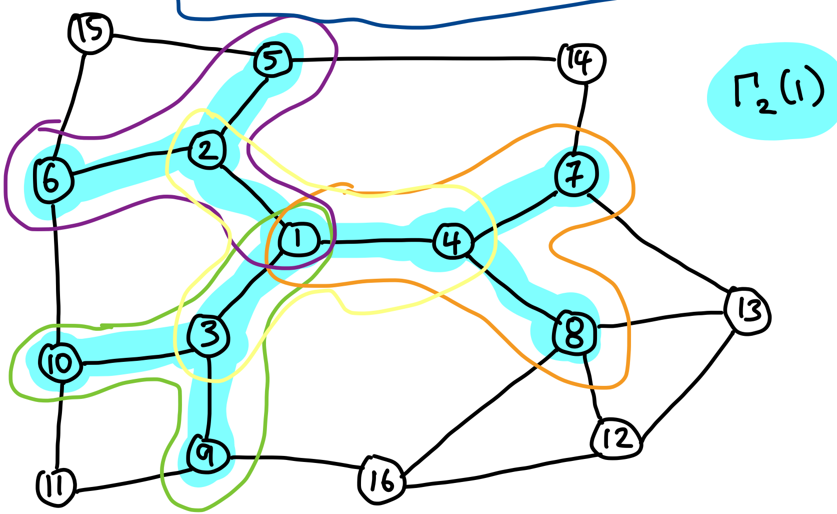


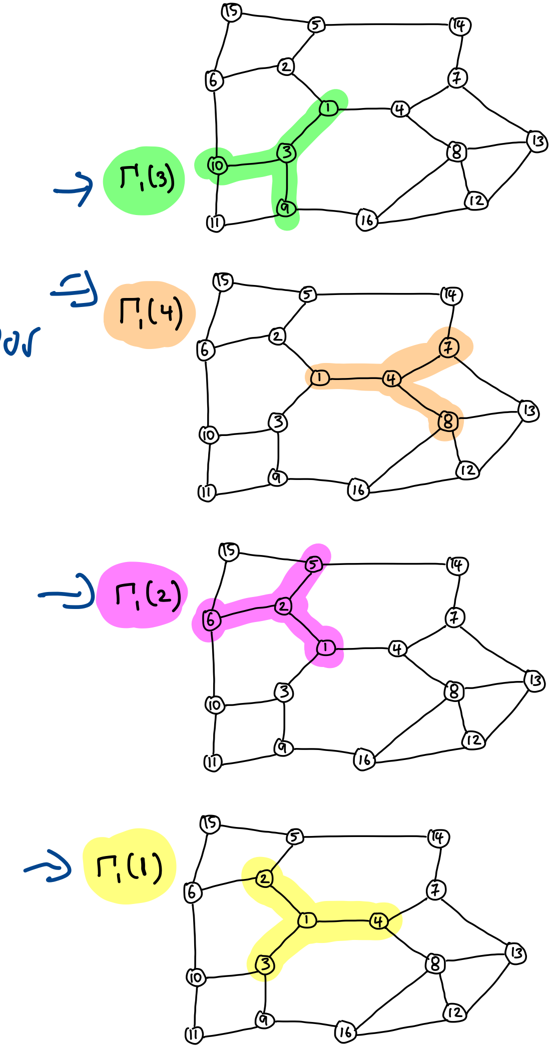
Neighborhood Covering Lemma. Let $G=(V,E)$ be a graph and $v \in V$ a vertex in G. For any distance $d \geq 2$, the distance d neighborhood of v is the union of v's neighbors distance $d-1$ neighborhoods.

Symbolically:

w is v's neighbor

$$\boxed{\Gamma_d(v) = \bigcup_{w \in \Gamma_1(v)} \Gamma_{d-1}(w)}$$



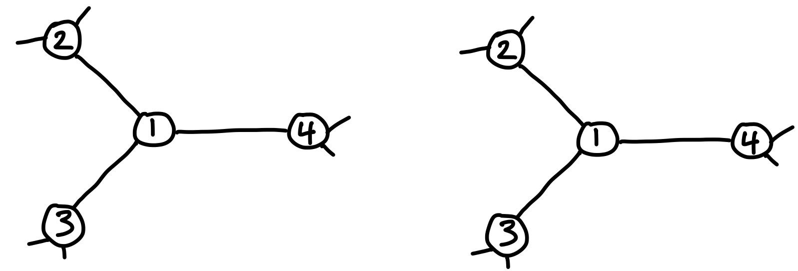


For Now. Take neighborhood covering lemma on faith

→ see notes for a proof

Want to show: state of each vertex v is determined by the initial state of vertices in $\Gamma_{r-1}(v)$. Formally:

Locality Lemma. Suppose Π is a PO protocol, $G = (V, E)$ is a graph together w/ local inputs for each vertex v. Then for every round r, and vertex v, the state of v in round r is determined by the initial state of $\Gamma_{r-1}(v)$.



Proof idea. The state of v in round 1 is determined only by v's local input

$\Rightarrow$ msgs sent in round 1 are functions of local inputs

State in round 2 is function of v's local input and msgs received in round 1

$\Rightarrow$ msgs sent by v in round 2 are functions of inputs of distance 1 neighborhoods

...

Proof. Want to show that for all vertices v and rounds r, v's state is determined by $\Gamma_{r-1}(v)$.

Argue by induction on r.

Base case: $r = 1$.

- initial state of v is determined by v's local input: hence $\Gamma_0(v)$.

$= \{v\}$

Inductive Step: $r-1 \Rightarrow r$

lemma applied to $r-1$

Suppose in round $r-1$, every u's state is determined by $\Gamma_{r-2}(u)$

Then msgs sent by each u in rnd $r-1$ are determined by $\Gamma_{r-2}(u)$

<u>PO protocol</u> consists of

Sets:

- Inputs = allowable local inputs
- States = allowable local states
- Outputs $\subseteq$ States halting states w/ outputs
- Messages = allowable messages

Functions (d = degree of node)

- <u>init</u>: Inputs $\to$ States

  determine initial state from initial input

- <u>send</u>: States $\to$ Messages^d

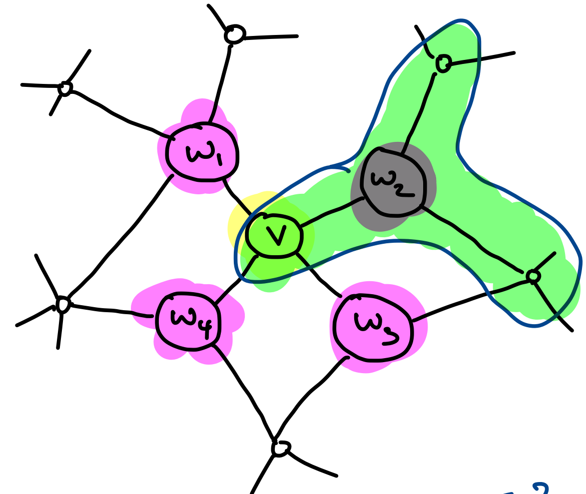
  determine the d messages to send to neighbors from current state

- <u>receive</u>: States $\times$ Messages^d $\to$ States

  determine how to update state from received messages

Inductive step continued...

Then msgs sent by each u in rnd $r-1$ are determined by $\Gamma_{r-2}(u)$



$r = 3$

$r - 1 = 2$

Consider vertex v w/ neighbors $w_1, w_2, \dots, w_k$ ($k = \deg(v)$)

Msg received from w_i in round $r-1$ is determined by $\Gamma_{r-2}(w_i)$

$\Rightarrow$ state in round r determined by v's state and msgs received in round $r-1$

$\Gamma_{r-2}(v) \cup \Gamma_{r-2}(w_1) \cup \Gamma_{r-2}(w_2) \cup \cdots \cup \Gamma_{r-2}(w_k) = \Gamma_{r-1}(v)$

$\Rightarrow$ v's state in round $r-1$ is determind by $\Gamma_{r-1}(v)$. //

A More Algorithmic View of Locality Lemma

In round 1, consider $\Gamma_{r-1}(v)$

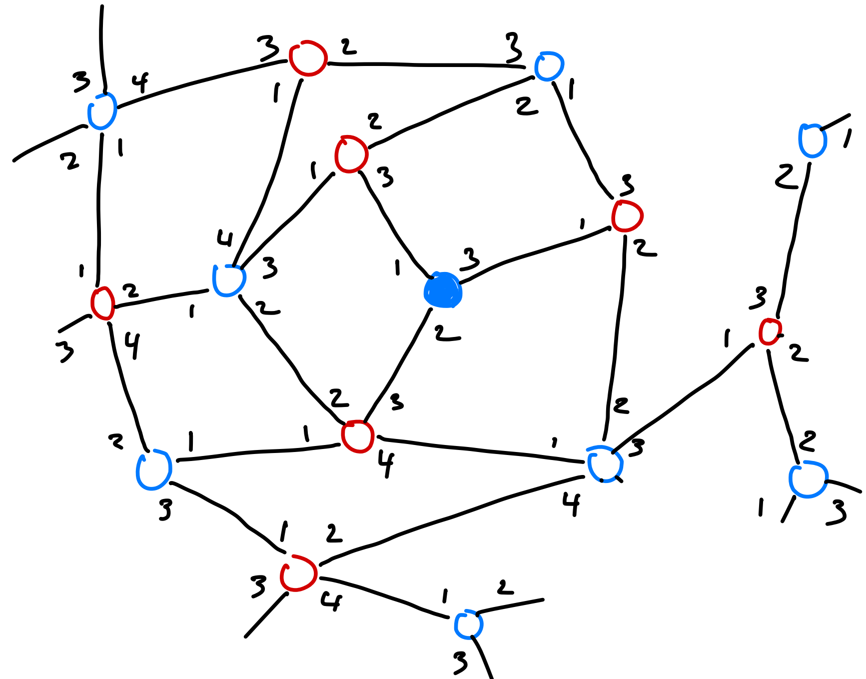
- compute all init. states
- compute all msgs
- determine all <u>received</u> msgs in $\Gamma_{r-2}(v)$

In round 2, consider $\Gamma_{r-2}(v)$

- compute all rnd 2 states
- compute all rnd 2 msgs
- find all <u>received</u> msgs in $\Gamma_{r-3}(v)$

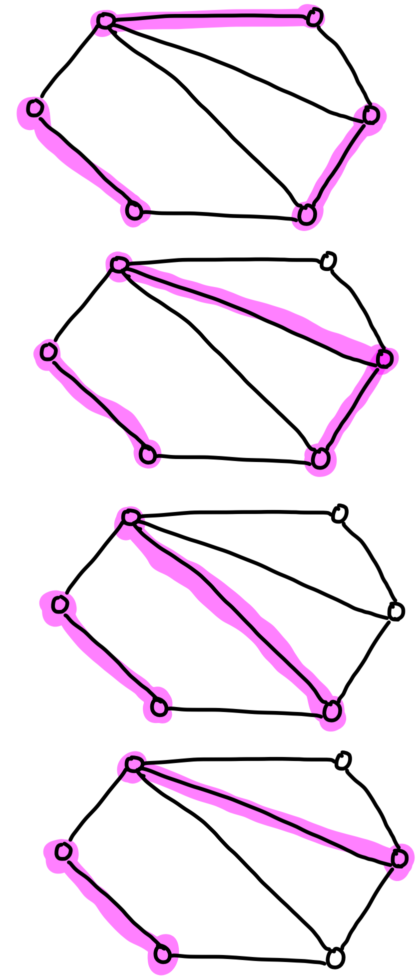
⋮

In round $r-1$, get all states / received msgs in $\Gamma_0(v) = \{v\}$.



Maximal Matchings

- $G = (V, E)$ a graph
- $M \subseteq E$ a set of edges is a <u>matching</u> if each vertex is incident to at most one edge in M
- A matching M is a <u>maximal matching</u> if there is no larger matchim M' that contains M
  - Equivalently, M is maximal if every $v \in V$ is either incident to an edge in M, or all of v's neighbors are incident to edges
  - informally: no edge can be added to M to result in a matching



Which are (maximal) matchings

Stable vs Maximal Matchings

- Consider context of SMP
  - PO network
  - 2 colored : blue nodes are students, red nodes are internships
- What if we are content to find a maximal matching between students and internships?

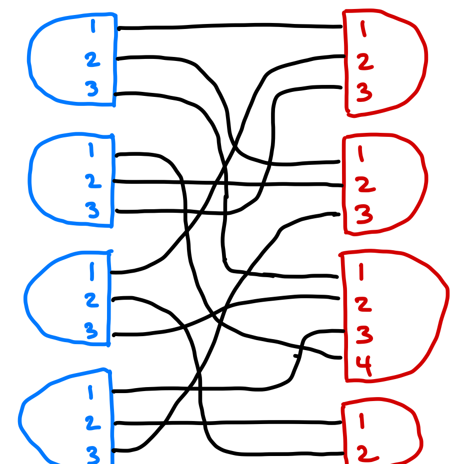
SM Instance

	1	2	3	4
Anna :	a	b	c	
Beck :	c	b	a	
Cameron :	a	d	c	
Daniel :	c	d	b	

	1	2	3	4
a :	A	C	B	
b :	A	B	D	
c :	A	C	D	B
d :	D	C		

$\Downarrow$

PO Network



Questions to consider

1. Are stable matchings always maximal matchings?
2. Are maximal matchings always stable?
3. Can maximal matchings be larger than a stable matching?
4. Can stable matchings be larger than maximal matchings

Big Question. Can maximal matchings be found efficiently?