Homework 3

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Question 1. So far, we have assumed that in an execution of a distributed protocol, all nodes simultaneously begin the execution in round 1. That is, we imagine that all nodes simultaneously wake in the same round and begin executing the protocol. In practice, however, nodes may be added to the network and activated at different times so that even if transitions between rounds are synchronized, the round numbers stored by individual nodes are not synchronized. For example, a network with three nodes may initially have "local" round numbers 1, 2, and 3. In the next round, they will transition to rounds 2, 3, and 4, respectively, and so on. In this question, you will consider the "simultaneous reset" problem, in which nodes must all simultaneously transition to round 1, regardless of their initial local round number. Thus, by using simultaneous reset, a network can simulate a protocol in the LOCAL/CONGEST/PO model without the assumption that all nodes simultaneously start in round 1.

To formalize things, suppose that initially in round 1, each node v starts at local round number $t_v \geq 1$. Each round, each node increments it local round counter, so that in rounds $1, 2, 3, \ldots$, a node v has local round counter $t_v, t_v + 1, t_v + 2, \ldots$. Crucially, the nodes do not have access to the global round number, and can only reference their local round count t_v. The `SIMULTANEOUS_RESET` problem requires that at some (global) round $r_0 \geq 1$, every node in the network simultaneously enters a state `RESET` at which time each node resets its local round clock t_v to 1. Thus, in rounds $r_0, r_0 + 1, r_0 + 2, \ldots$ all nodes v will have $t_v = 1, 2, 3, \ldots$. Describe a protocol that solves the `SIMULTANEOUS_RESET` problem in the CONGEST model in $r_0 = O(D)$ rounds.

Question 2. Let $G = (V, E)$ be a graph, and $w : E \to \mathbf{R}^+$ an assignment of positive weights to each edge in the graph. The ***weighted length*** of a path $P = (v_0, v_1, v_2, \ldots, v_k)$ in G is the sum of the weights of the edges in P. That is, $w(P) = w(v_0, v_1) + w(v_1, v_2) + \cdots + w(v_{k-1}, v_k)$. In the weighted single-source shortest-paths (SSSP) problem, `w-SSSP`, all nodes receive the ID of the same node u as input, and each node v should output its weighted distance from u—i.e., the path from v to u of minimal weighted length.

(a) Assuming that each edge weight can be encoded in $O(\log n)$ bits, devise a protocol for `w-SSSP` in the CONGEST model that runs in $O(n)$ rounds (where n is the number of nodes in the network).

(b) For any given n, describe a network with n nodes with diameter $D = O(1)$ for which your algorithm takes $\approx n$ rounds to terminate despite the network's small diameter.

(Hint: for part (a), show that the Bellman-Ford algorithm can be implemented in the CONGEST model. For termination, you may assume that all nodes know the size n of the network.)

Question 3. Let $G = (V, E)$ be a graph. A ***proper*** k ***coloring*** of G is an assignment of a "color" (number) from the range $1, 2, \ldots, k$ to each vertex such that no two neighboring vertices are assigned the same color. In the distributed problem `k-COLORING`, each node receives no local input (other than its ID and port numbering); each node v should output a number from the range $1, 2, \ldots, k$ such that if a vertex v outputs o_v, none of its neighbors output o_v. Suppose G is a graph with n vertices and maximum degree (at most) Δ, for some integer $\Delta \geq 1$.

(a) Show that G admits a proper coloring with $k = \Delta + 1$ colors.

(b) For any k, give an example of a graph G_k with maximum degree Δ such that G_k does not admit a proper Δ coloring.

(c) Suppose we are given a CONGEST protocol Π for the maximal independent set (MIS) problem that runs in $T(n)$ rounds on all networks with n nodes. Devise a CONGEST protocol that uses Π as a sub-routine to compute a property $\Delta + 1$ coloring of G in $O(\Delta T(n))$ rounds.

(Hint: for part (a), devise a greedy algorithm that is guaranteed to produce a proper $\Delta + 1$ coloring of G. For part (c), consider the similarity between proper coloring and MIS: in MIS, if a node v outputs 1 (v is in the MIS), then none of its neighbors can output 1. Similarly in a proper coloring, if v outputs i, then none of v's neighbors can output i.)