Homework 1

COSC 373, Spring 2022

Will Rosenbaum
Department of Computer Science
Amherst College
Updated: February 20, 2022

Quesion 1. In *College Admissions and the Stability of Marriage*, Gale and Shapley consider stable matching instances with n students and n internships in which every student ranks every internship, and vice versa. They claim that the Gale-Shapley algorithm terminates after at most $n^2 - 2n + 2$ rounds of proposals.

(a) Construct an instance (i.e. preferences) with 3 students and 3 internships such that the Gale-Shapley algorithm only terminates after $5(= 3^2 - 2 \cdot 3 + 2)$ application rounds. Be sure to record (1) the preference lists, (2) the applications made and rejected in each round, and (3) the stable matching found by the algorithm.

(b) (*Challenge*) Generalize your construction from part (a) to give preferences for any n that require $n^2 - 2n + 2$ application rounds for the Gale-Shapley algorithm to terminate.

Question 2. Consider again stable matching instances with complete preference lists: there are n students and n internships, every student ranks every internship, and vice versa. Suppose all students share the same ranking of internships, and all internships have the same ranking of students.

(a) What is the stable matching found by the Gale-Shapley algorithm for such instances? How many application rounds are required until the algorithm terminates?

(b) Argue that the matching found in part (a) is the *unique* stable matching for these instances.

Hint: Suppose the students are $S = \{s_1, s_2, \ldots, s_n\}$ and the internships are $U = \{u_1, u_2, \ldots, u_n\}$. Without loss of generality, you may assume that every student ranks the internships in order $u_1, u_2, \ldots, u_n$ and every internship ranks the students in order $s_1, s_2, \ldots, s_n$.

Question 3. Suppose $G = (V, E)$ is a graph. A *matching* M is a set of edges in G such each vertex $v \in V$ is incident to at most one edge $e \in M$. We say that v is *matched* in M if there is some edge (v, u) incident to M contained in M. A *maximal matching* is a matching in which each vertex $v \in V$ is either matched, or all of v's neighbors are matched.*

* Thus, no edge from G can be added to M to result in a larger matching.

(a) Devise a (centralized) greedy algorithm that takes as input a graph $G = (V, E)$ and computes a maximal matching M. (For this part, it is sufficient to provide a high level description of your procedure.)

(b) Suppose G has n vertices and m edges, and we are given an adjacency array representation of G. That is, we a provided an array containing G's vertices, and each vertex v has an associated array of v's neighbors. Use "big O" notation to describe the running time of your procedure as a function of n and m. (You may assume that the adjacency array of each vertex v stores both each neighbor w of v, as well as the index of v in w's adjacency array.)

(c) Given a matching M in a graph, let $|M|$ denote the size of the matching—i.e. the number of edges contained in M. Give an example of a graph G and two *maximal* matchings M_1 and M_2 such that $|M_1| = 2\,|M_2|$. (Hint: G need not be a large graph!)

(d) (*Challenge*) Is it possible that a graph G can have two maximal matching M_1 and M_2 with $|M_1| > 2\,|M_2|$?

Question 4. Let $G = (V, E)$ be a graph. Given two vertices $u, v \in V$, the *distance* between u and v, denoted $\mathrm{dist}(u, v)$ is defined to be the length of the shortest path from u to v, or $\mathrm{dist}(u, v) = \infty$ if there is no path from u to v. Prove that dist satisfies the *triangle inequality*: if $u, v, w \in V$ are vertices, then

$$\mathrm{dist}(u, w) \leq \mathrm{dist}(u, v) + \mathrm{dist}(v, w).$$