Lecture 34: P and NP

COSC 311 *Algorithms*, Fall 2022

Announcement

Job Candidate Talk **TODAY**

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> *Using Simultaneous Multithreading to Support Real-Time Scheduling*

- 4:00 in SCCE A131
- Refreshments at 3:30 in SCCE C209

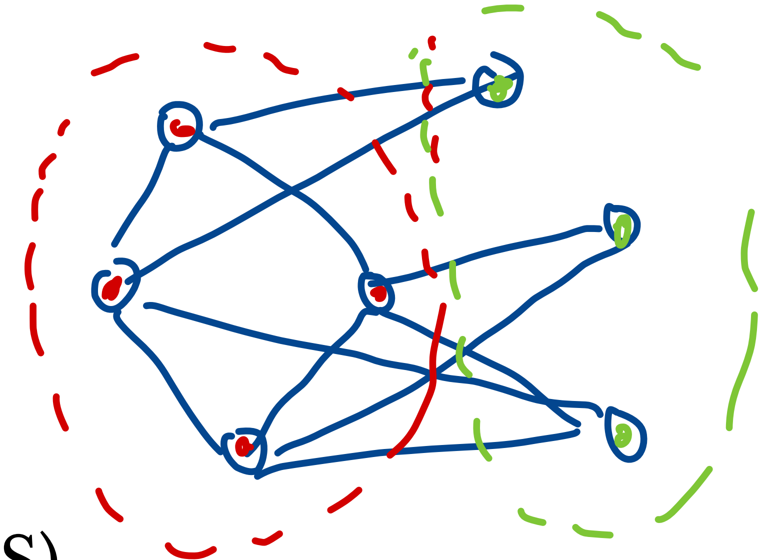
Homework 6

Posted soon...

...not to be turned in!

- solution posted later this week

Last Time



Two Problems:

- Minimum Vertex Cover (MVC)
- Maximum Indpendent Set (MaxIS)

Polynomial-time reductions between them:

- MVC $\leq_P$ MaxIS
- MaxIS $\leq_P$ MVC

Consequence

- MVC can be solved efficiently $\iff$ MaxIS can be solved efficiently

Today

1. Decision Problems
2. The Classes P and NP

A Technicality

Objective. Understand *relationships* between computational problems.

Technical issue. Desired outputs for different problems can be vastly different:

- matching
- independent set
- spanning tree
- ...

Convenience. Focus on **decision problems**:

- output is “yes”/”no”

MVC vs VC

Minimum Vertex Cover (MVC)

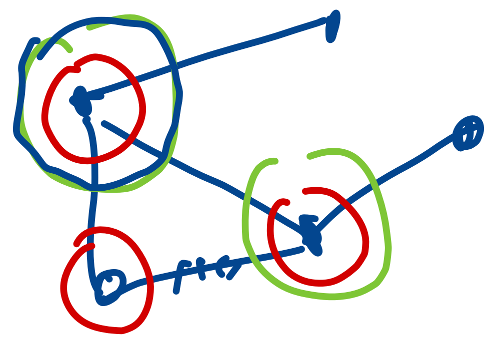
- *Input:* Graph G
- *Output:* A *vertex cover* C of smallest possible size

Vertex Cover (VC)

Input: Graph G, number k

Output:

- "yes" if G has a vertex cover of size k
- "no" otherwise

$G =$ 

$VC(G,3)$ = yes!

$VC(G,2)$ = yes

$VC(G,1)$ = no

MaxIS vs IS

Maximum Independent Set (MaxIS)

Input: Graph G

Output: an indpendent set of the largest possible size

Independent Set (IS)

Input: Graph G, number k

Output:

- "yes" if G has an indpendent set of size k
- "no" otherwise

Complexity of Decision Problems

Goal. Classify (decision) problems according to their relative *complexities*:

Polynomial $\rightarrow$ $O(N^c)$ for some c

- which problems can be solved $\boxed{\text{efficiently?}}$
- which problems $\boxed{\text{cannot be solved efficiently?}}$
- which problems can be reduced to other problems?

e.g. $\Omega(2^N)$

The Class P

Definition. The class **P** consists of all decision problems that can be solved in polynomial time.

- P = “polynomial time”
- a problem A is in P if there is an algorithm that given any instance X of A
  - the algorithm correctly outputs “yes”/”no”
  - the running time is $O(N^c)$ for some constant c, N = size of input

Which Problems Are In P?

$B = 2^{\log B}$ $\boxed{\log B}$

EXP

Knapsack : can get value of V
and weight $\leq B$

n elements, $b_1, \ldots b_n$ $O(n \cdot B)$

Not polynomial b/c B exp. in log B
" size of rep. of B.

Min. Spanning Tree. $O(m \log n)$ – Prim

Decision variant: "is there a spanning tree of weight $\leq k$?"

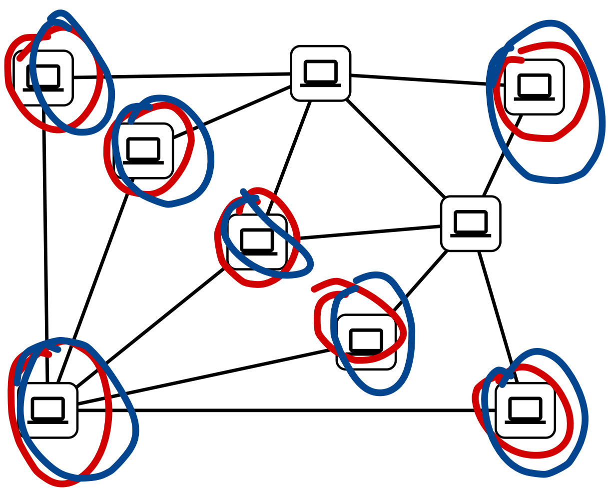
Text Partition Problem from HW.

The Class NP

Verifying Output

Consider IS(G, k):

- "yes" if G has an indpendent set of size k
- "no" if G does not have an indpendent set of size k



Ask for the IS of size k

→ verify by checking if is an IS.

→ $O(m)$ if m edges in graph

Question. How could you be convinced that IS(G, k) = "yes?"

Check for edges in IS, if edge is found "reject"

NP, Informally

NP = "nondeterministic polynomial time"

Informal Definition. The class **NP** consists of decision problems whose solution can be *verified* in polynomial time.

IS

G, k

Setup

- *A* is a decision problem, *X* an instance (input) of *A*
- If *X* is a "yes" instance, there should be some way to convince me this is the case
- If *X* is a "no" instance, there should be no way to convince me *X* is a "yes" instance

Verifier

Definition. Given a decision problem A, a **verifier** for A is a polynomial time algorithm verify(X, C) that takes as input

Instance
e.g. G, k

Purported
I.S.

- an *instance* X of A, and
- a *certificate* C (size polynomial in size of X)

and returns a value "accept" or "reject," subject to two conditions:

1. **completeness** if X is a "yes" instance, then there exists a certificate C such that verify(X, C) returns "accept"
2. **soundness** if X is a "no" instance, then for every certificate C, verify(X, C) returns "reject"

A Verifier for IS

Consider IS(G, k):

- "yes" if G has an indpendent set of size k
- "no" if G does not have an indpendent set of size k

What is a verifier for IS?

- what should be the certificate?
- how do we verify a certificate?

set of k vert. C in G (purported IS)

for each vertex v in C, check if any of v's neighbors are in C → if so reject

accept otherwise

NP, Formally

Definition. The class **NP** consists of all decision problems that admit a polynomial time verifier.

NP, Formally

Definition. The class **NP** consists of all decision problems that admit a polynomial time verifier.

- By previous example, IS is in NP

Conceptually NP can be thought of the class of *puzzles*

- a puzzle may be hard to solve
- you can easily verify if you (or someone else) solved the puzzle

P vs NP

Open Question. Is there any problem in NP that is not in P?

Informal statement. Are there problems that are hard to solve, but whose solutions are easy to verify?

- one of deepest mathematical challenges of our time

Activity

Which of the following problems are in NP:

1. BipartiteMatching(G, k)
2. NoFlow(G, k)
3. GeneralizedChess(n, C)
4. BooleanSatisfiability$(\varphi(x_1, x_2, \ldots, x_n))$

Bipartite Matching

Question. Is BipartiteMatching in NP?

Decision: G — does G have a matching of size k?

Yes: C: a matching of size k

$\underline{(v_1, w_1)}, \underline{(v_2, w_2)}, \ldots, \underline{(v_k, w_k)}$

Verify:

(1) check these are edges in G

(2) check for no repeated vtx.

More Generally

If a problem A is in P, then A is in NP:

- $\text{P} \subseteq \text{NP}$

Why?

Verify:

- ignore certificate
- solve problem
- return "accept"/"reject" depending on if output of orig. prob is "yes"/"no"

NoFlow

Question. Is NoFlow in NP?

Observation

Even if did not know about the Ford-Fulkerson MaxFlow algorithm, we could still identify NoFlow is in NP.

How?

GeneralizedChess

Question. Is GeneralizedChess in NP?

Chess Remarks

A feature of chess games:

- a game may last exponentially many rounds in the size of the board
- therefore: winning strategy might require exponential time to describe/verify

Fact. GeneralizedChess *requires* exponential time to solve (in n).

Consequence. Showing GeneralizedChess is in NP would imply that $P \neq NP$.

Boolean Satisfiability

Question. Is BooleanSatisfiability in NP?

Next Time

- NP Completeness: characterizing the “hardest” problems in NP