Lecture 32: Bipartite Matchings and Reductions

COSC 311 *Algorithms*, Fall 2022

Overview

1. Reductions
2. Maximum Bipartite Matchings
3. Reduction to Maximum Flow

Big Picture, So Far

Efficient algorithms for many problems:

- sorting
- graph problems
  - shortest paths
  - Eulerian circuits
  - minimum spanning trees
- interval scheduling
- sequence alignment
- stable matching

All solved in $O(N^2)$ time (N = size of instance)

A Question

What algorithmic problems *cannot* be solved efficiently?

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What algorithmic problems *cannot* be solved efficiently?

- What do we mean by "efficiently?"
- How could we show that *no algorithm* can solve a problem efficiently?
    - What is an "algorithm," anyway?

Last Unit of the Semester

Algorithmic Reality

For many practical problems:

1. no efficient algorithm is known
2. no proof that there isn't an efficient algorithm

What do we do in this situation?

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What do we do in this situation?

Reductions & NP Completeness

- focus on *relationships* between problems
  - what does it mean for problem A to be no harder/easier than problem B?

Algorithm Life

Observation. Algorithm design is challenging.

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Lifestyle Choice. Avoid designing new algorithms (when you can get away with it).

How?

Combine previous algorithms

Find relationships between new and old problems.

Algorithm Life

Observation. Algorithm design is challenging.

Lifestyle Choice. Avoid designing new algorithms (when you can get away with it).

How?

Idea. Given a new problem A, *transform* it into a problem B you already know how to solve!

Example from homework: Scheduling contractors with bids

Bids from contractors

- interval of time
- cover total time for work

node in graph

Bid → 

Reductions

scheduling problem

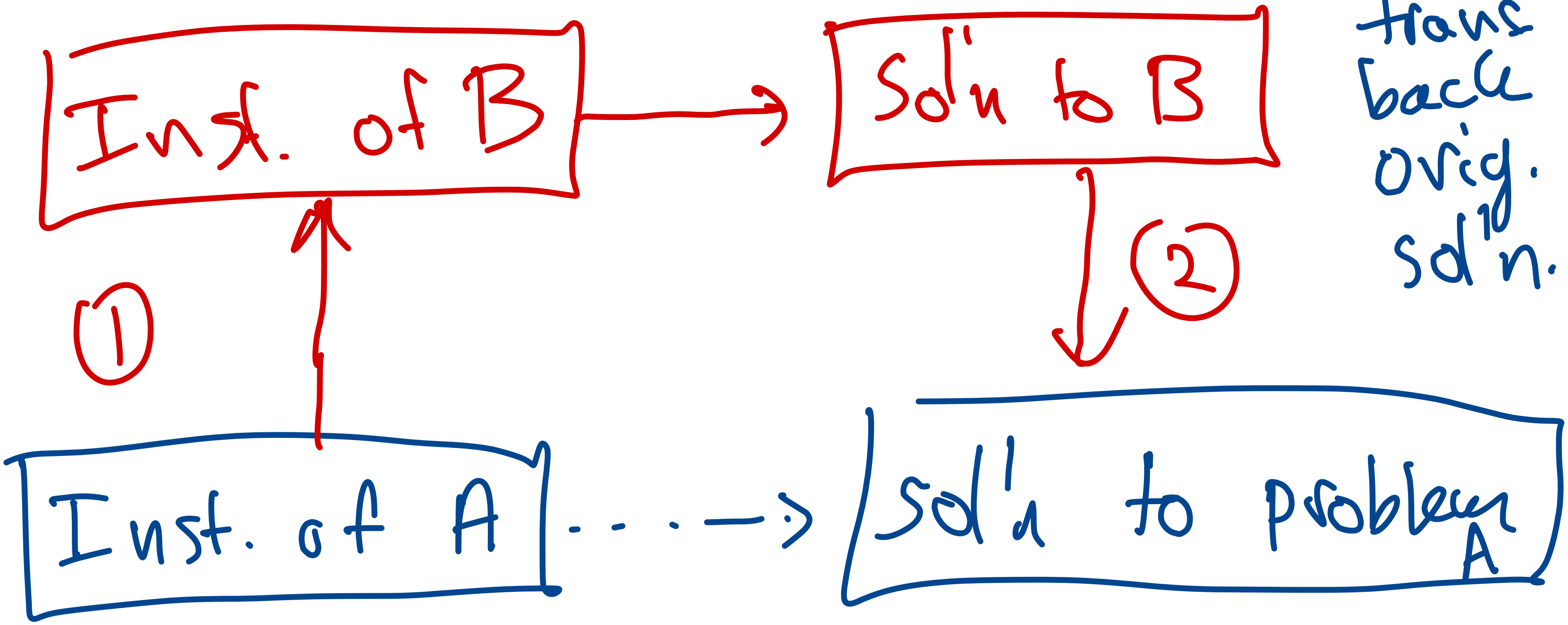
shortest paths

Properties of nice transformations:

1. transforming instances of $\boxed{A}$ to instances of $\boxed{B}$ can be done *efficiently*
2. solution to B instance can be *efficiently* transformed back to solution for A instance

Given n shortest path ↓ trans back to orig. sol'n.

1 + 2 = **reduction** from problem A to problem B



Reductions

Properties of nice transformations:

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$1 + 2 =$ **reduction** from problem A to problem B

Coarse notion of efficiency:

- transformations can be done in time $O(N^c)$ for some constant c ($N =$ input size)

In this case reduction is **polynomial time** reduction

- write $A \leq_P B$ if polynomial time reduction from A to B

Practical Value of Reductions

If $\boxed{A \le_P B,}$ then:

- any efficient solution to B gives an efficient solution to A
- an improved algorithm for B may give an improved algorithm for A

New Challenge

- Given a problem A, solve it by reducing to another problem B that you already have an algorithm for

Internship Assignments, Again

In a small world...

- Three students: a, b, c
- Three internships: X, Y, Z

Students/Internships have *acceptability criteria* (not preferences)

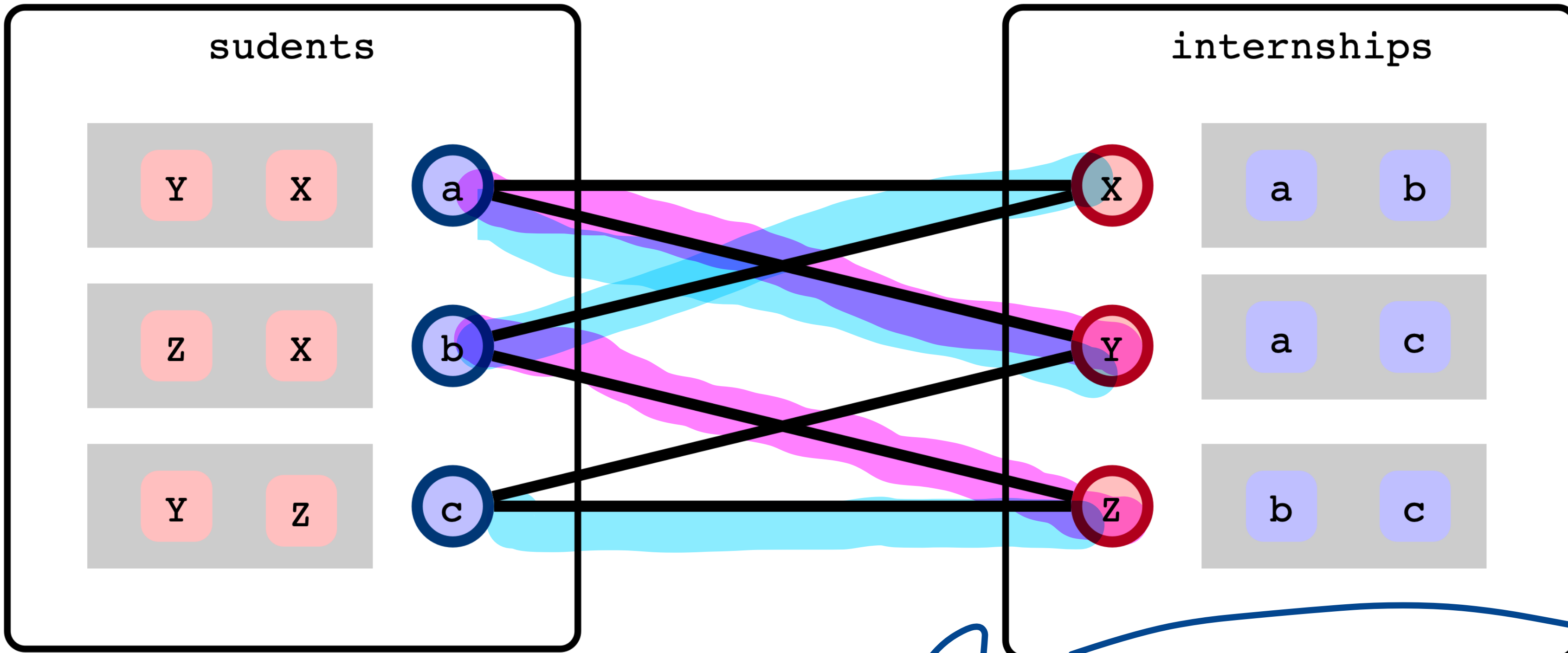
- $a : \boxed{Y}, X$
- $b : Z, X$
- $c : Y, Z$

- $X : a, b$
- $Y : \boxed{a}, c$
- $Z : b, c$

Symmetric

How to match students an internships?

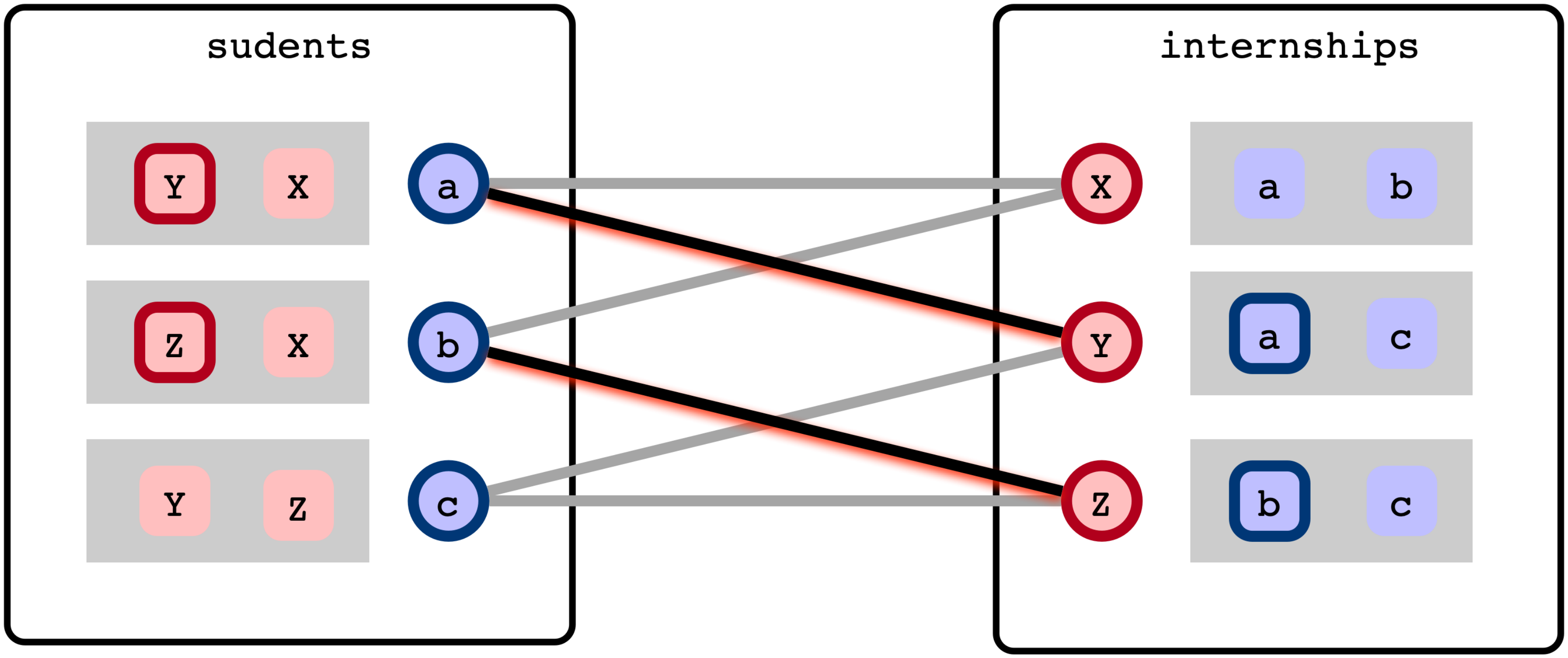
Viewed as a Graph



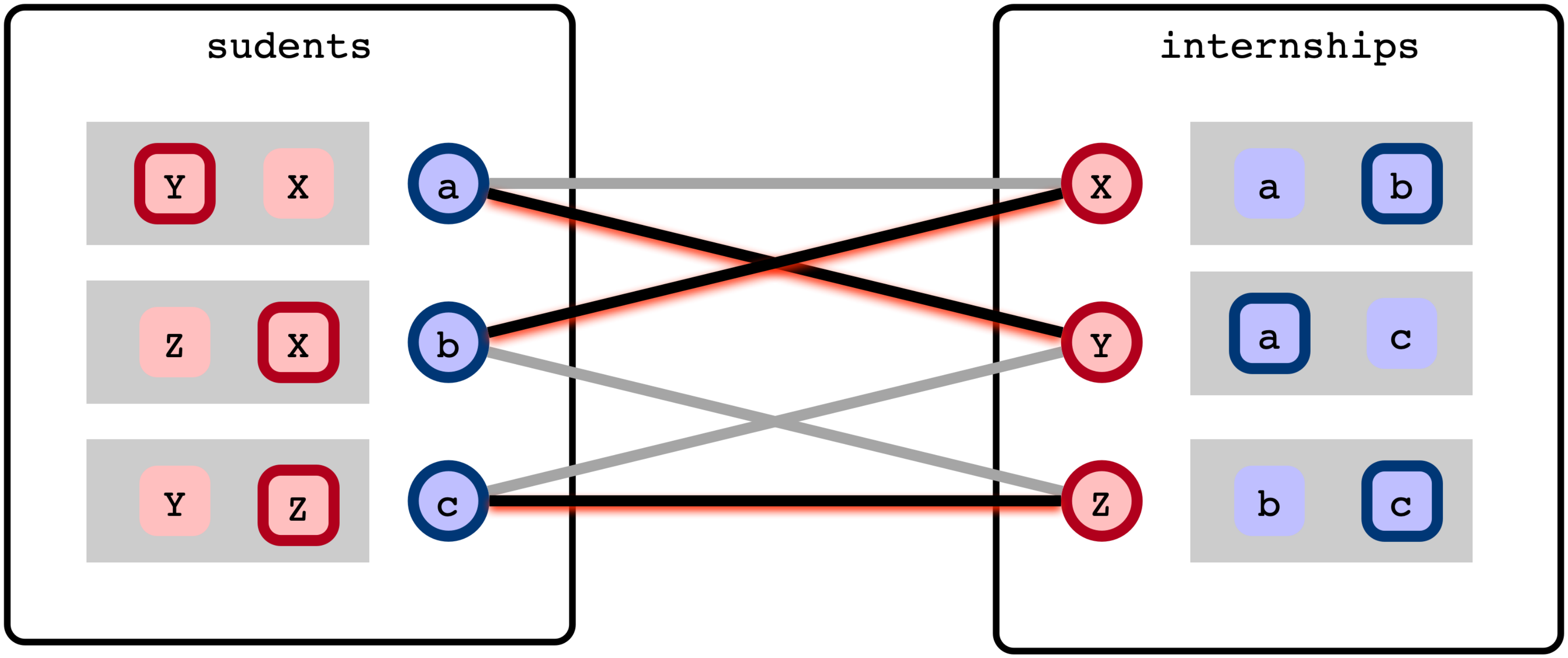
Pink matching matches 2, can't match more

blue matches everyone

Not Great Matching



Best Matching

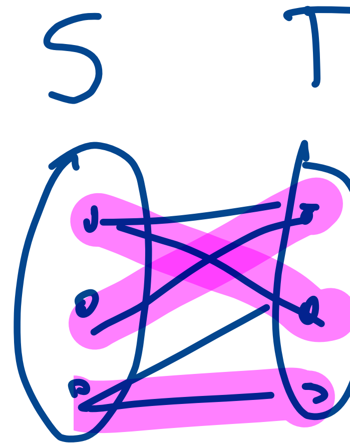


Maximum Bipartite Matching (MBM)

students internships

Input:

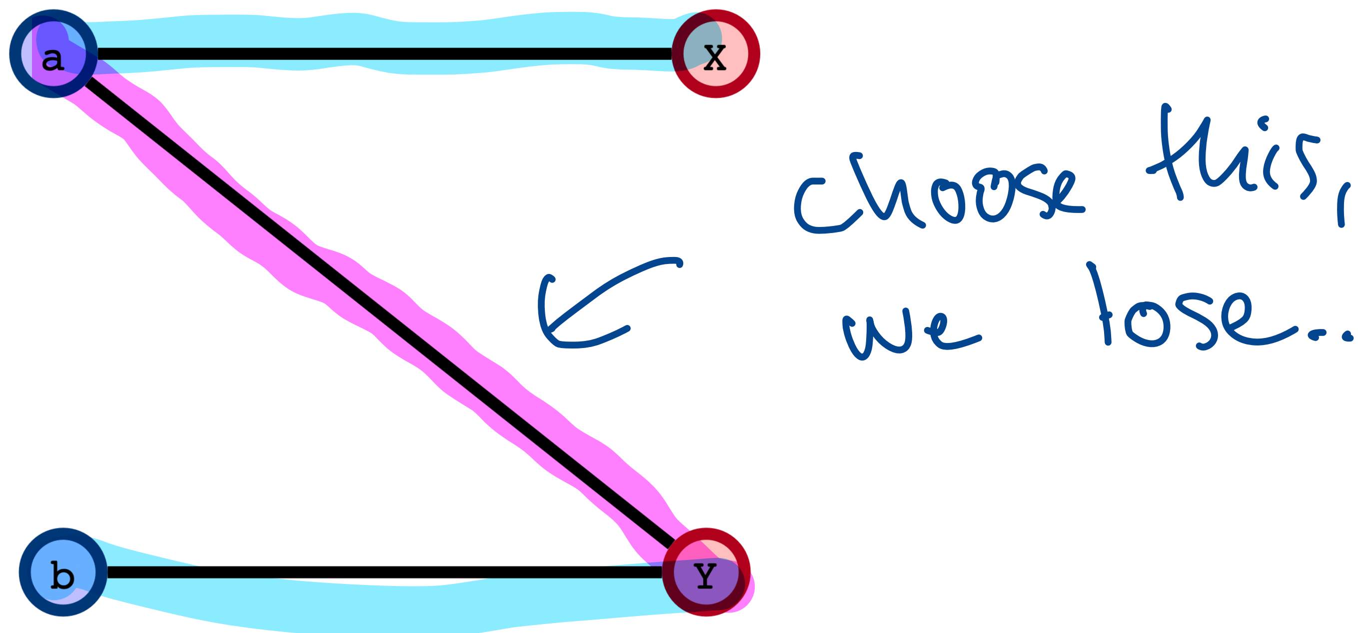
- bipartite graph $G = (V, E)$
- V partitioned into two disjoint sets, S, T
- All edges are pairs (s, t) with $s \in S, t \in T$



Output:

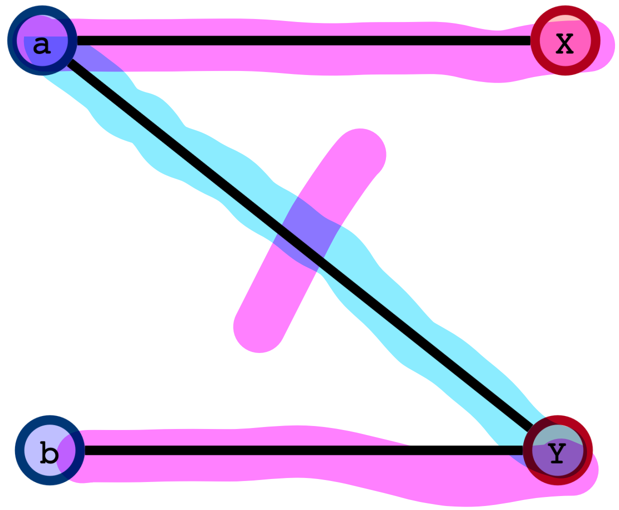
- a matching $M = \{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}$
  - each (s_i, t_i) is an edge in G ← acceptable to each other
  - each $s \in S, t \in T$ appears in at most one pair
- M is a **maximum matching**: there is no matching of size $\ell > k$

Simplest Interesting Example



Does greedily adding edges to form a matching always result in a maximum matching?

Greedy Doesn't Work



Issue: choosing an edge greedily/prematurely may *block* other edges that could result in a larger matching

- does this sound familiar?

Network flow

Adapting to Our New Lifestyle

Don’t design a new algorithm...

...instead make a reduction to...

Adapting to Our New Lifestyle

Don't design a new algorithm...

...instead make a reduction to...

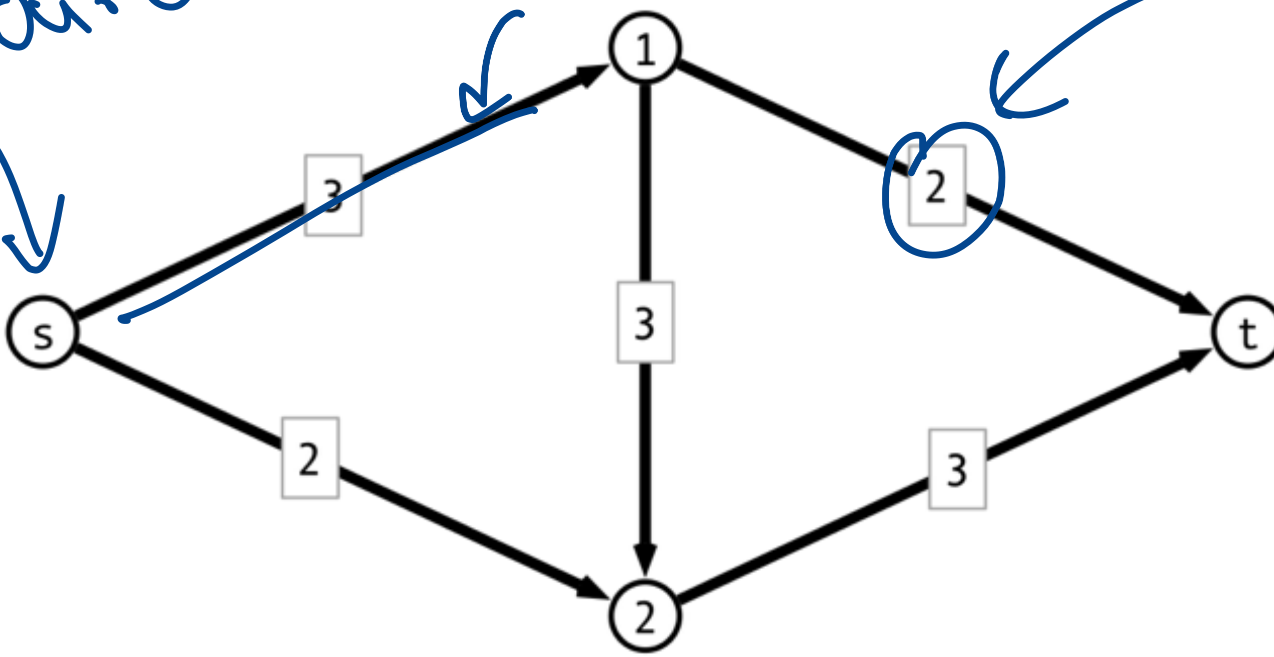
Maximum Flow!

source

Directed edges

capacities

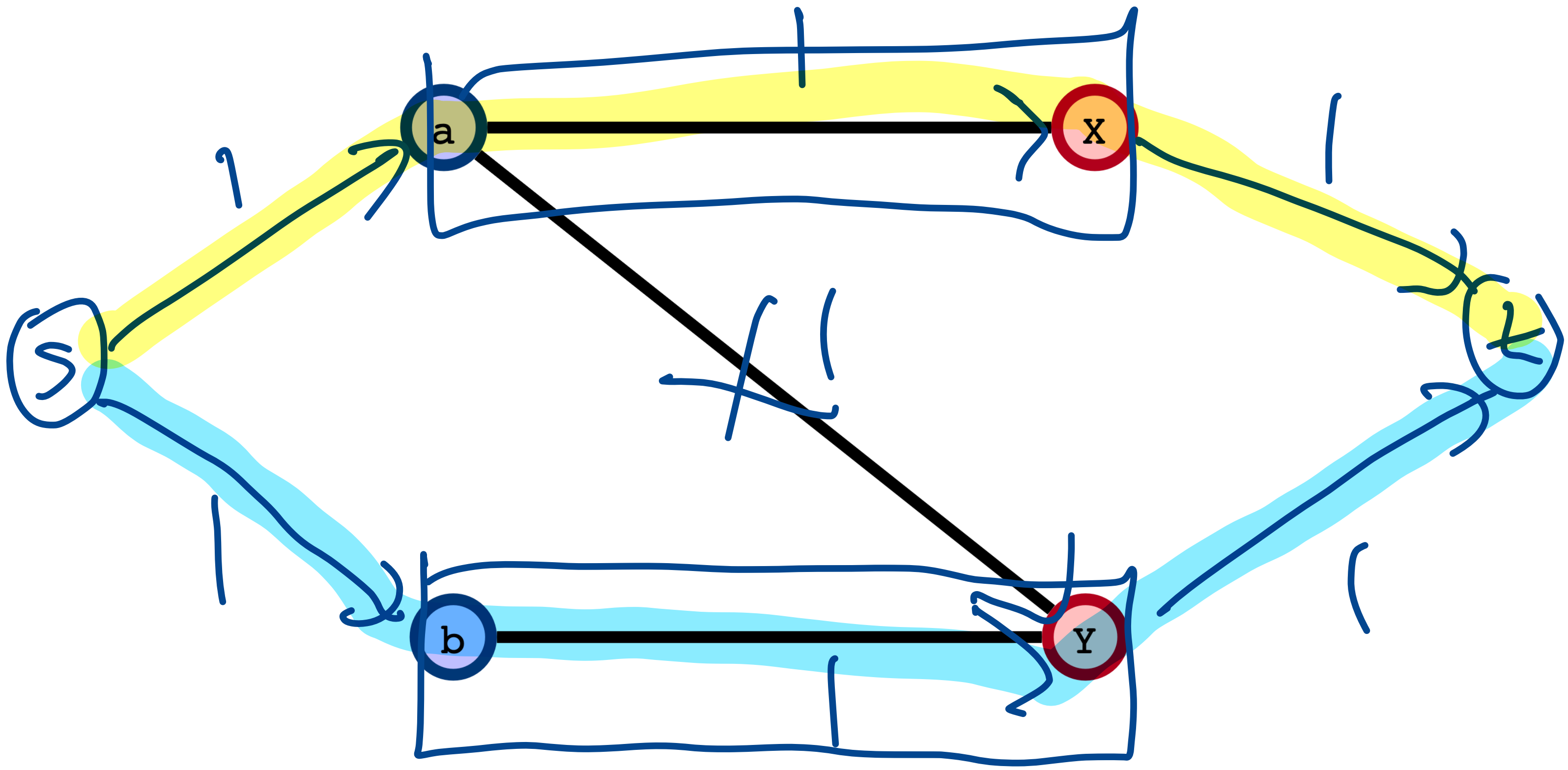
sink



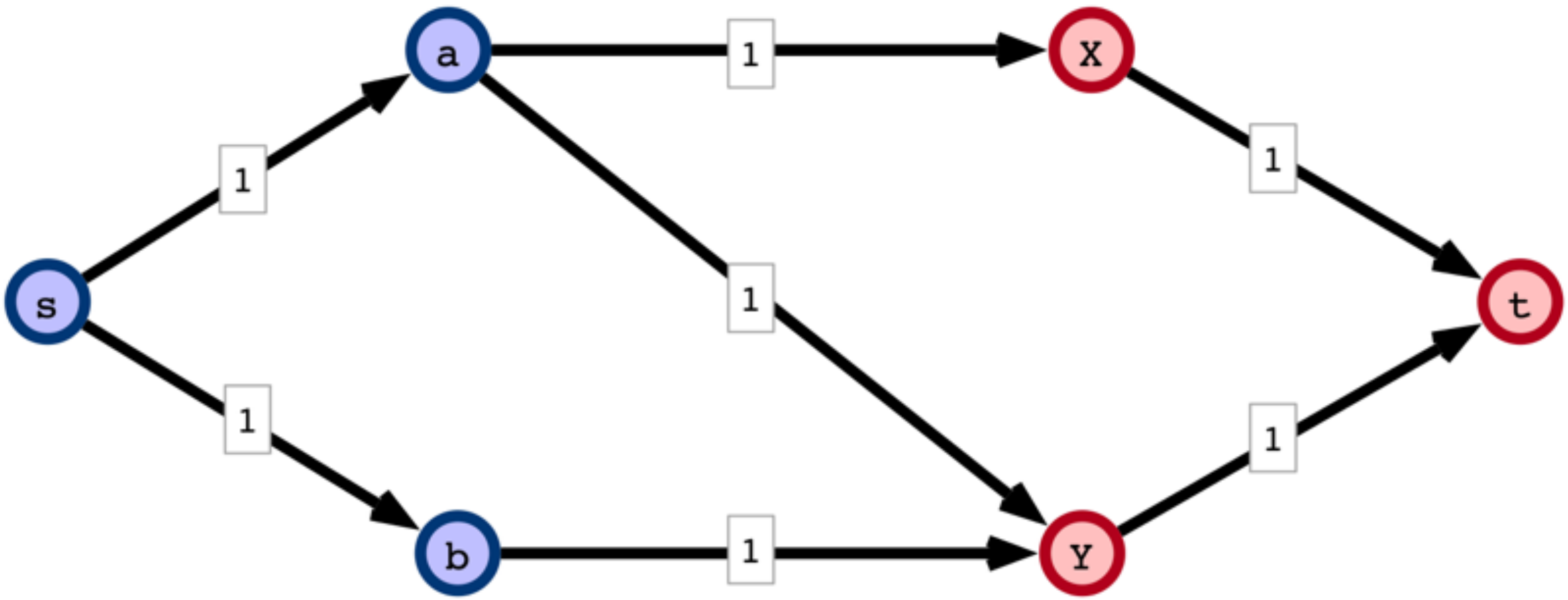
Find flow that maximizes amount of fluid from s to t

Reduction to Max-Flow

How to turn MBM into a flow problem?



A Max-Flow Instance



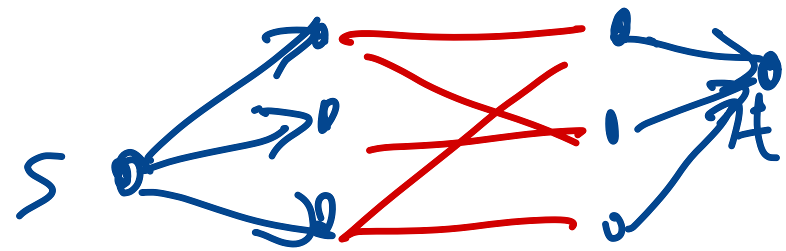
Formalizing the Reduction

Input:

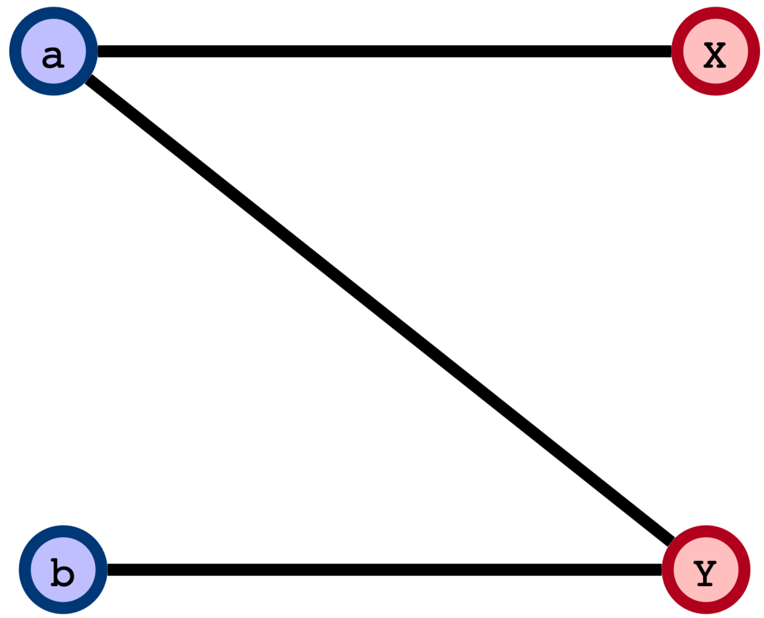
- Bipartite graph $G = (V, E)$ with $V = S \cup T$

Output:

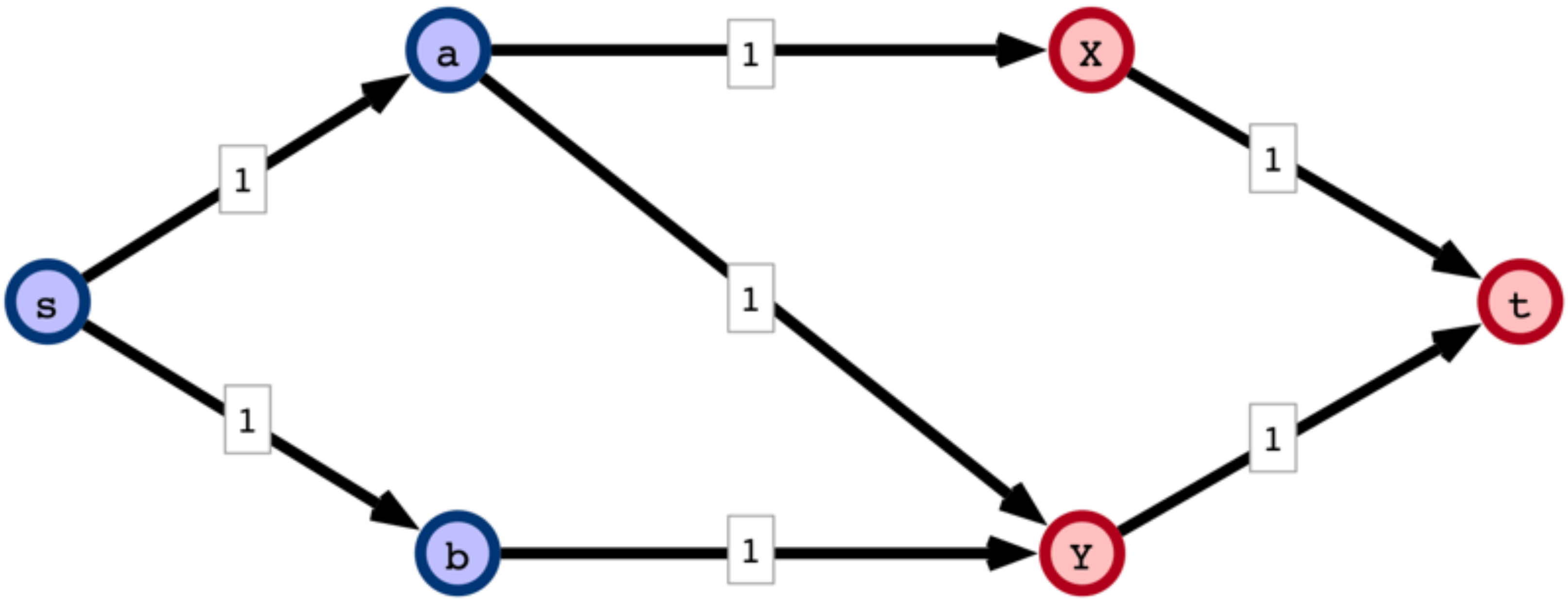
- Directed graph $G' = (V', E')$
- $V' = V \cup \{s, t\}$ — add source & sink
- For E':
  - direct all edges $(s_i, t_j) \in E$ from S to T
  - add edges (s, s_i) for all $s_i \in S$
  - add edges (t_j, t) for all $t_j \in T$
  - all capacities are 1



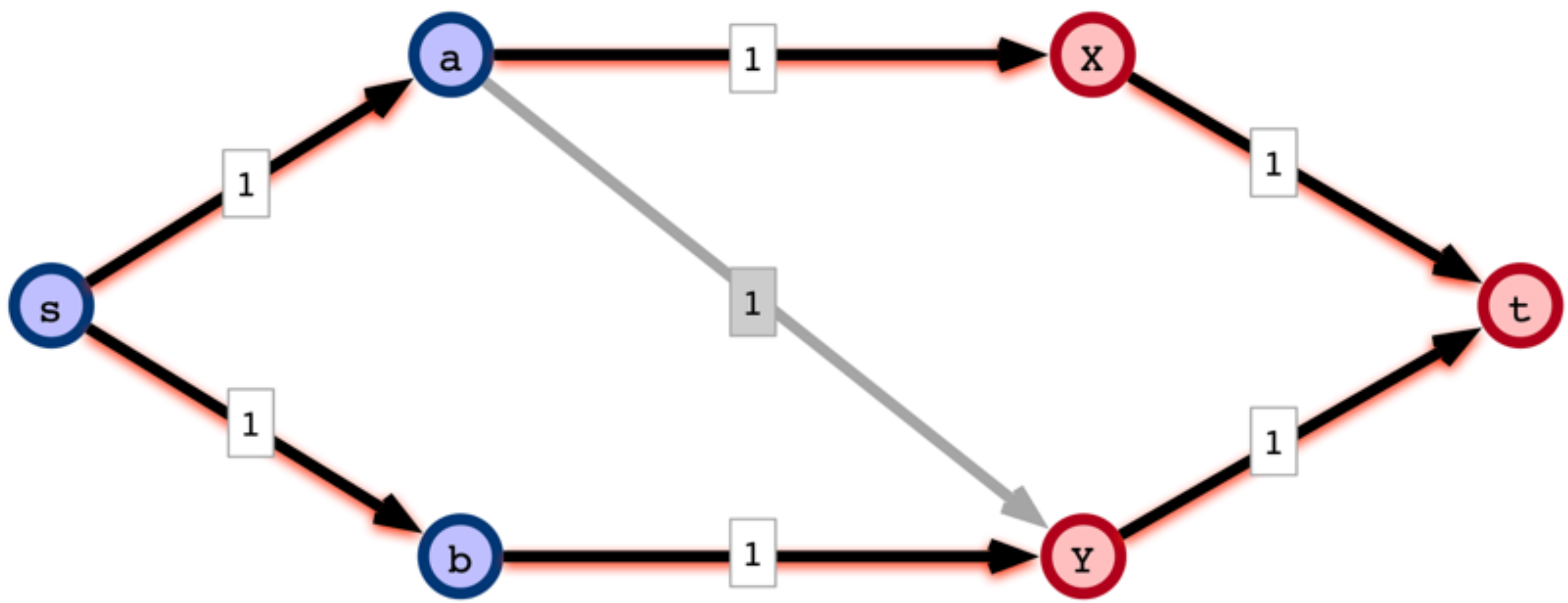
I Start with Bipartite Graph



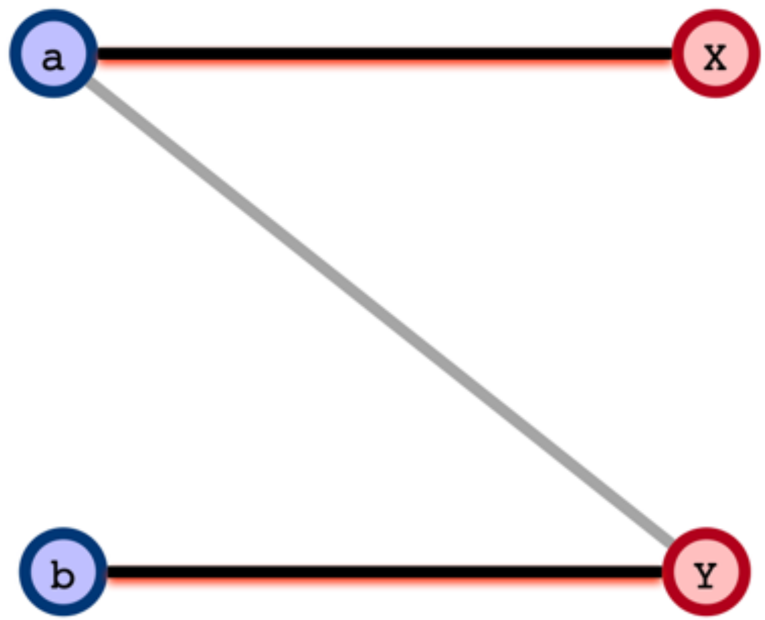
II Form Max Flow Instance



III Solve Max Flow



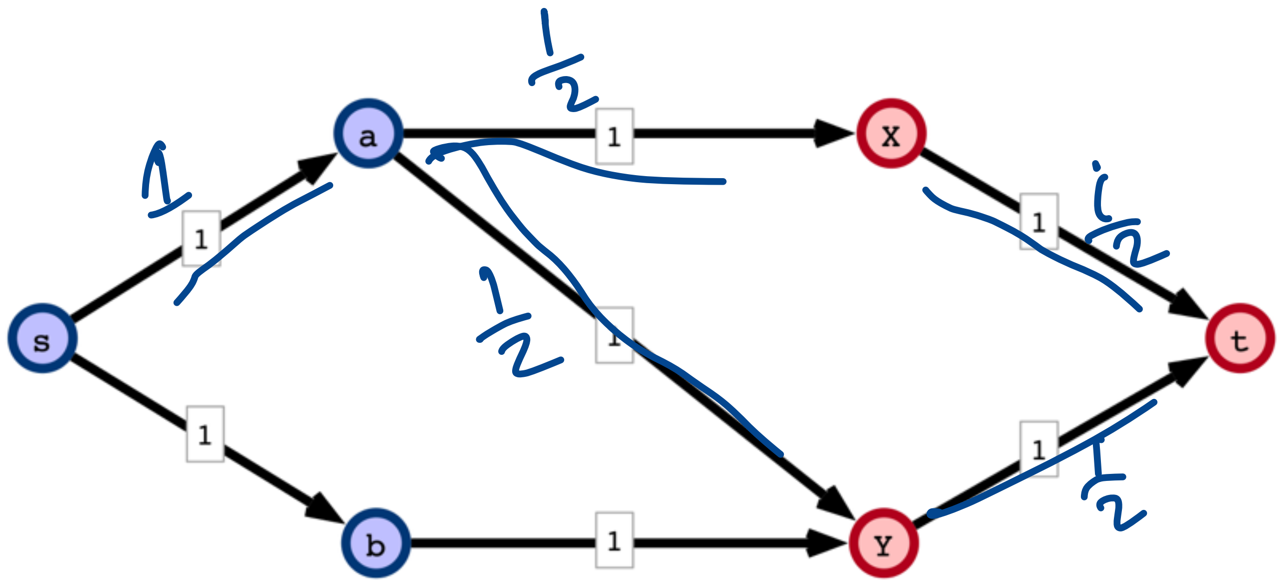
IV Form Matching from Flow



A Technicality

Assume all flows are **integral flows**:

- amount of flow across each edge is an integer

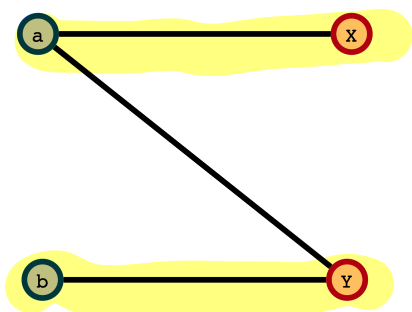


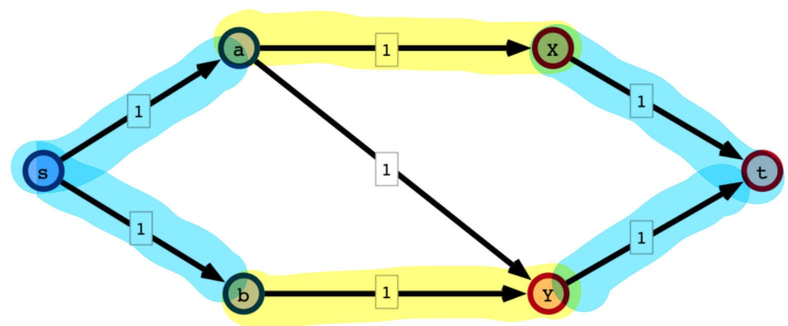
Note. Ford-Fulkerson always gives an integral flow.

flow 1 gets included in matching
flow 0 does not

Claim 1

If G has a matching of size k, then G' admits a flow f of value k.





For each (s_i, t_i) in matching route 1 unit of flow along

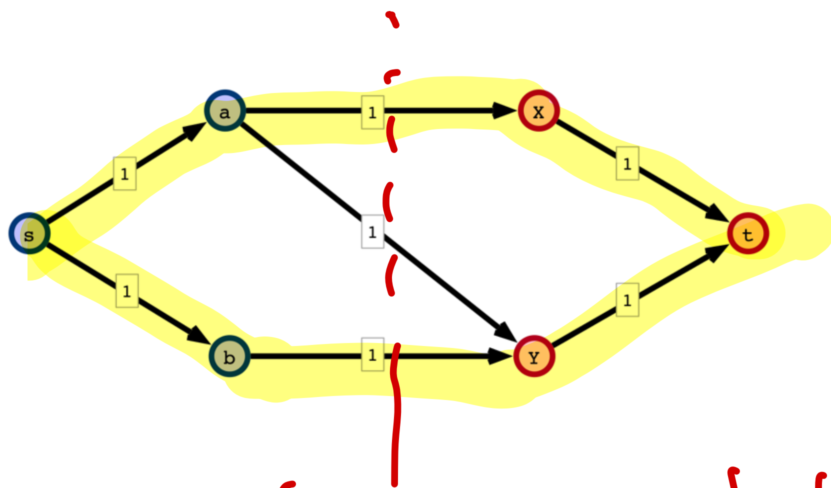
s, s_i, t_i, t

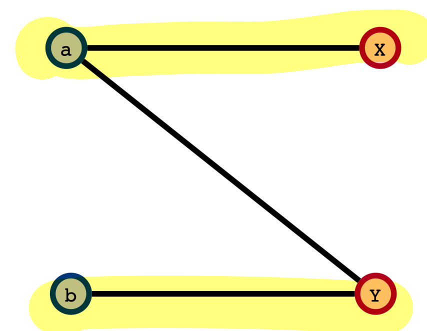
1 value of flow per matching edge $\Rightarrow$ k value flow overall.

Claim 2

0 or 1 since capacities are all 1

If G' admits an integral flow of value k, then G has a matching of size k.





Consider matching edges w/ pos. flow.

these must form a matching ...

Implications

1. Value of the maximum flow in G' equals the size maximum matching in G.
2. Given a maximum (integral) flow in G', we can find a maximum matching in G

Conclusions

- Found a *reduction* from Maximum Bipartite Matching to Max Flow
- Any (integral) Max Flow algorithm can be used to solve MBM
  - any improvement in Max Flow algorithms ~~will~~ might yield a corresponding improvement in MBM solution
- We showed MBM $\leq_P$ Max Flow
  - MBM is "no harder" than Max Flow
  - Max Flow is "no easier" than MBM

Next Time

Another view of $\leq_P$:

- suppose A is a "hard" problem
- we show $A \leq_P B$
- then we've established that B is no easier than A
  - any efficient solution to B would imply an efficient solution to A