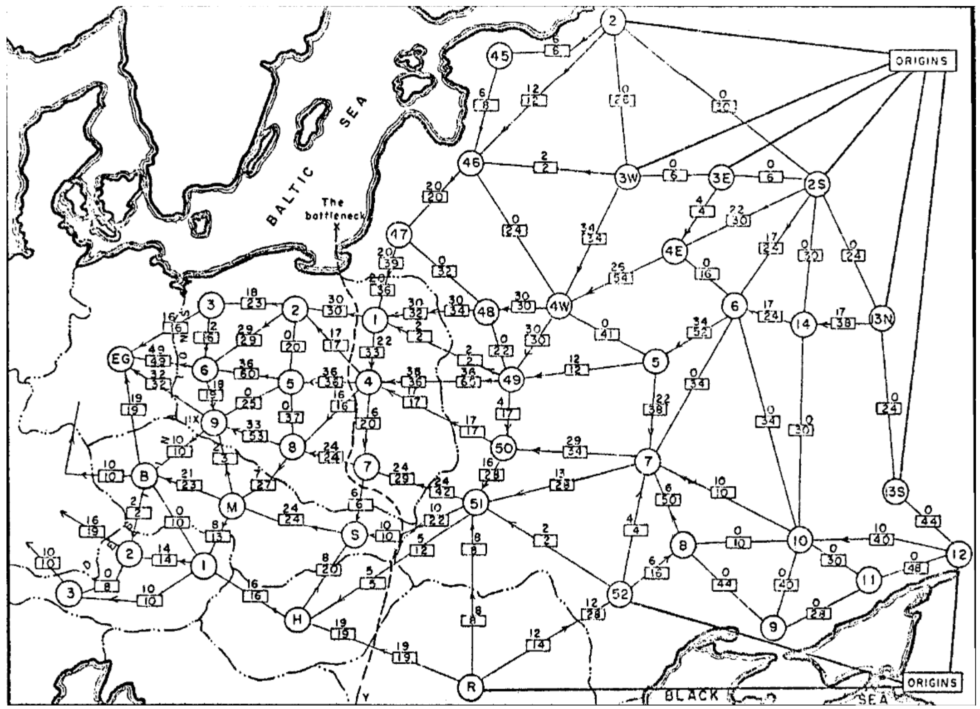
Lecture 29: Network Flow II



COSC 311 *Algorithms,* Fall 2022

Last Time

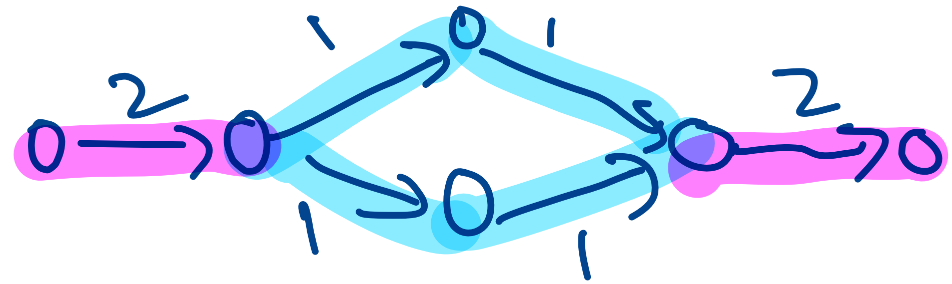
Network Flow

A new interpretation of directed graphs:

- network of (directional) pipes
- weights are *capacities*
  - how much fluid can flow through piper per time
- designated *source node s*
  - all edges directed away from *s*
- designated *sink* or *destination node t*
  - all edges directed towards *t*

Question. How much fluid be routed from *s* to *t* per unit time?

Flows, Formally



Setup.

- $G = (V, E)$ a directed graph, s, t source and sink
- $c(u, v)$ is capacity of edge (u, v)

f(e) = how much flow crosses e per time

Flows. An **s-t flow** f is a function $f : E \to \mathbf{R}^+$ satisfying:

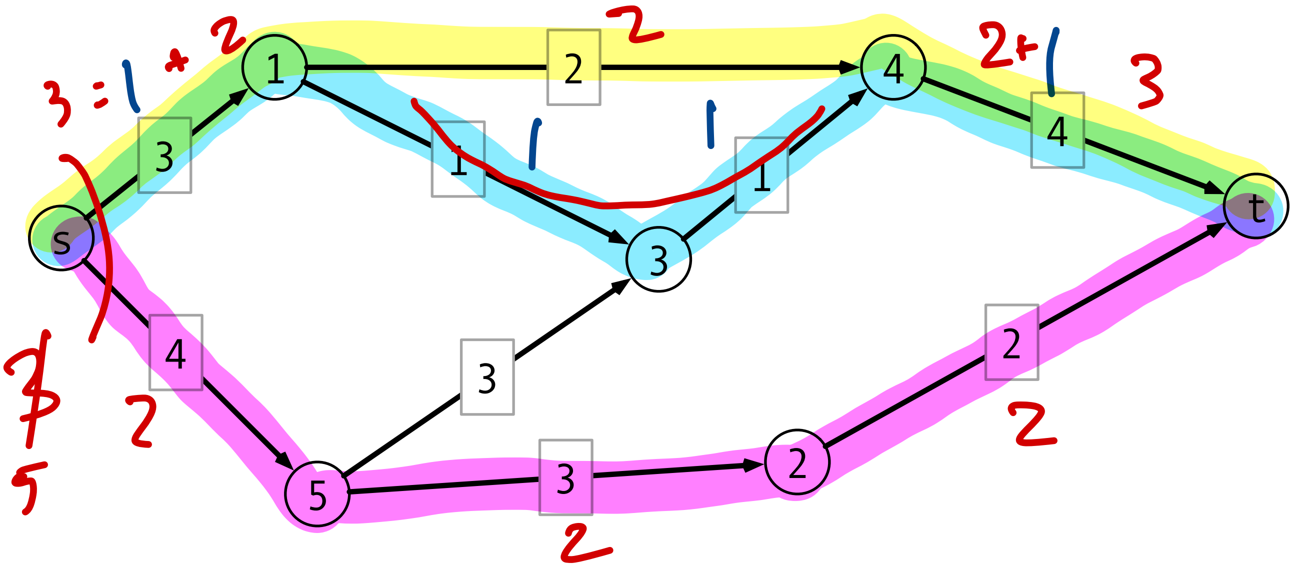
1. *capacity constraints:* for each edge e, $f(e) \leq c(e)$
2. *conservation:* for every vertex $v \neq s, t$, flow into v = flow out of v:
   - $\sum_{x \to v} f(x, v) = \sum_{v \to y} f(v, y)$

flow into v — flow out of v

The **value** of the flow f is $\text{val}(f) = \sum_{s \to v} f(s, v)$

amount of flow leaving source

Flow Example



e	s,1	s,5	1,3	1,4	2,t	3,4	4,t	5,3	5,2
f(e)	1	2	1	0	2	1	1	0	2

Max Flow Problem

Input.

- weighted directed graph $G = (V, E)$
  - weights = edge capacities > 0
- source s, sink t
  - all edges oriented out of s
  - all edges oriented into t

Output.

- flow f of maximum value
  - $\text{val}(f) = \sum_{s \to v} f(s, v)$

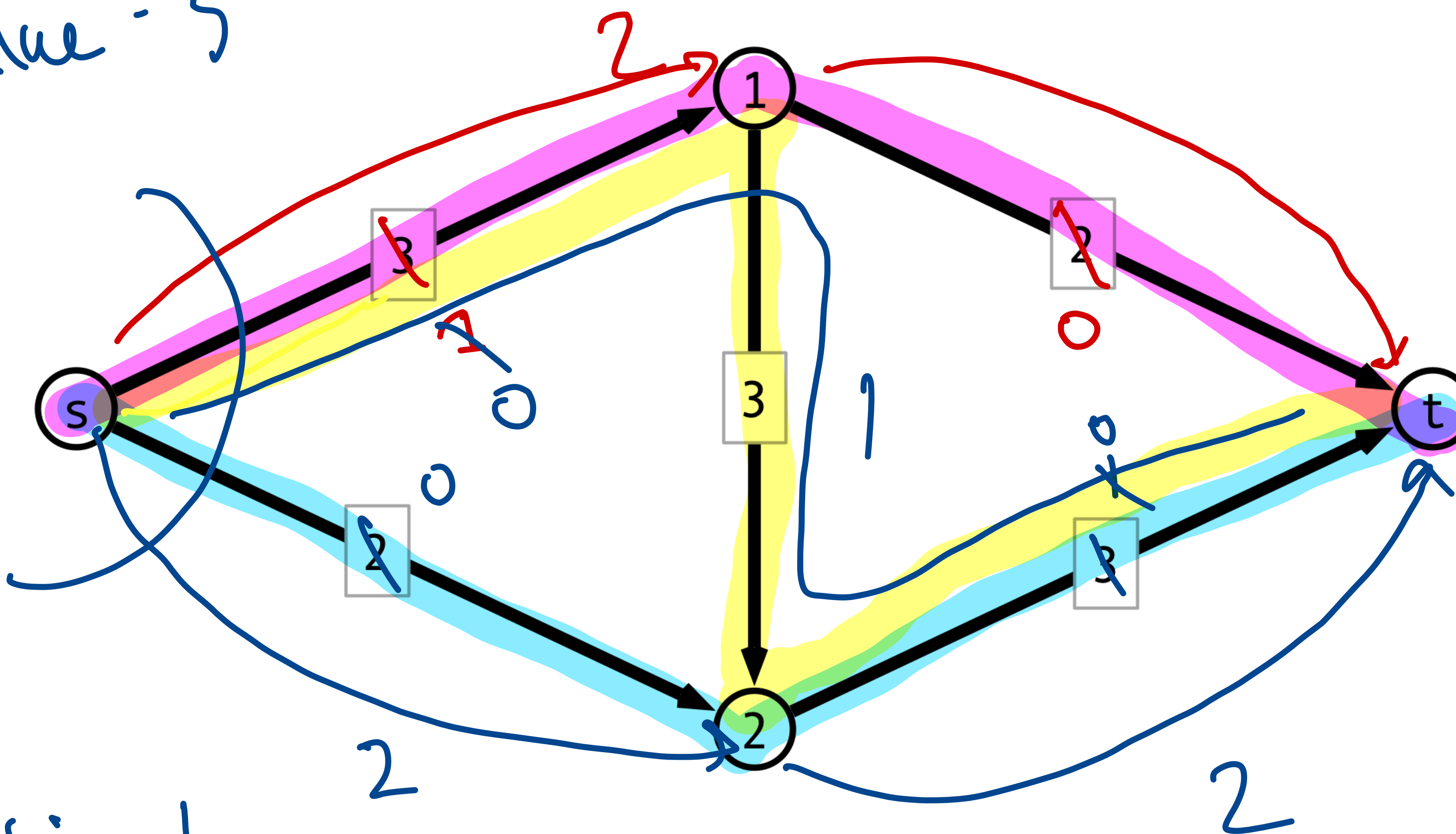
A Simple Greedy Strategy

Repeat until done:

1. find an "unsaturated" path P from s to t
2. find minimum (remaining) capacity b along P
3. route b units of flow along P

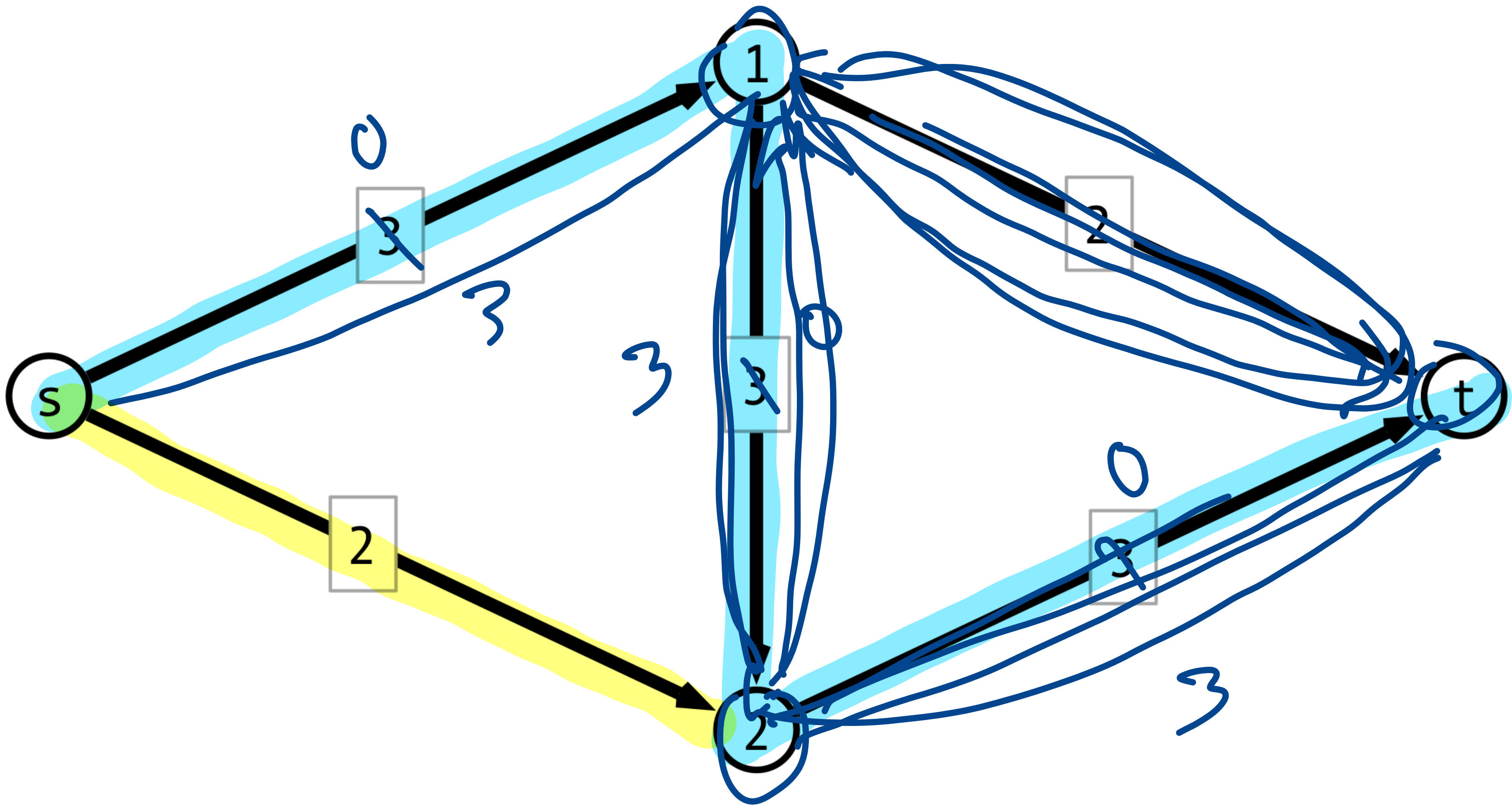
Greedy Approach Example

value = 5



maximal b/c all source edges saturated

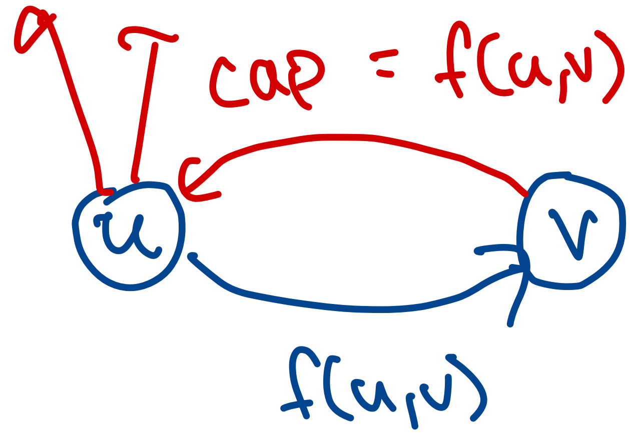
Choosing Different First Path



Greedy Issue

Flow along P may block other viable paths

Question. How to fix this?

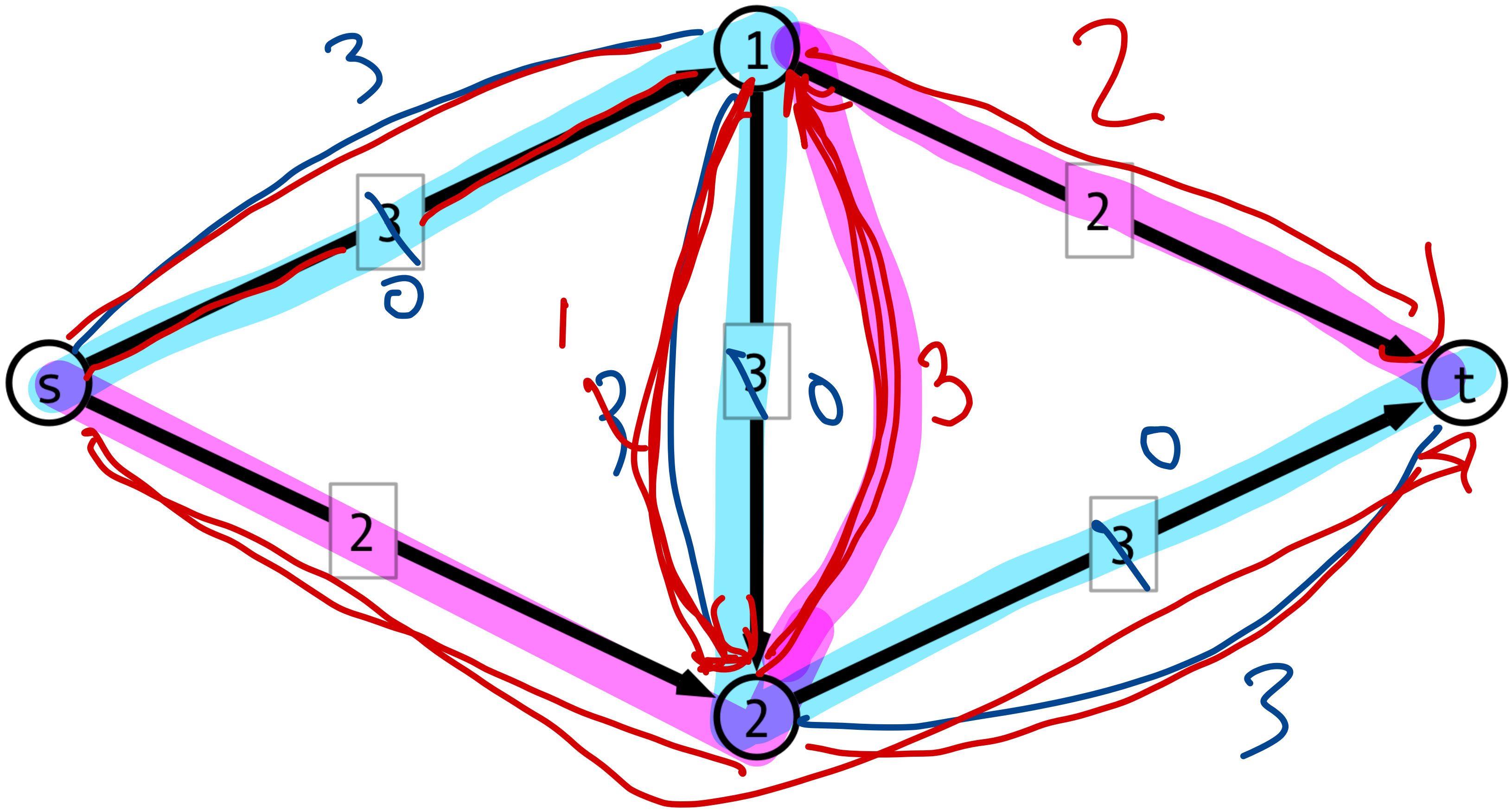


Augmenting Paths

Idea. Add “undo” feature for each edge

- if f routes $f(u, v) \leq c(u, v)$ flow from u to v, add reverse edge (v, u) with capacity $c(v, u) = f(u, v)$
- using (v, u) corresponds to “pushing back” flow from (u, v)
- if an alternate route for this flow can be found, then more flow can be routed through u

Pushing Back Example



The Residual Graph

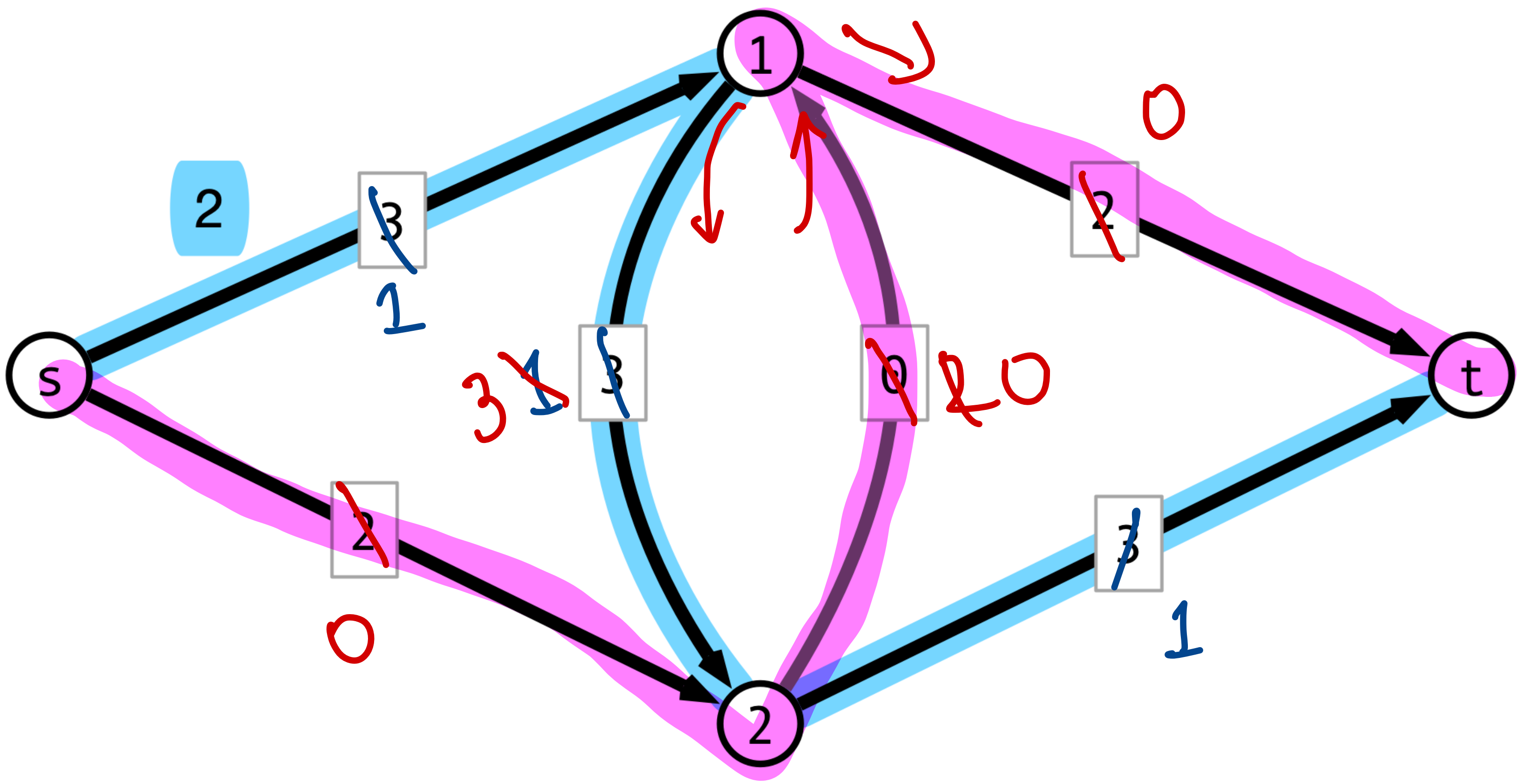
- $G = (V, E)$ original graph
- f a flow on G

Residual graph $G_f = (V_f, E_f)$

- vertex set $V_f = V$
- for each $(u, v) \in E$, add (v, u) to E_f
  - (u, v) is *forward edge*
  - (v, u) is *backward edge*
- in G_f capacity of (u, v) is:
  - $c(u, v) - f(u, v)$ if $(u, v) \in E$ (forward edge)
  - $f(v, u)$ if $(v, u) \in E$ (backward edge)

remaining capacity if f(u,v) units of flow cross (u,v)

Residual Graph Example



e	s,1	s,2	1,t	2,t
f	2	2	2	2

Ford-Fulkerson Algorithm

orig graph

Very high level

0

1. Initialize residual graph, flow f
2. While there is a path from s to t in residual graph do:
   - find path P from s to t
     - ignore edges with capacity 0
   - $b \leftarrow$ minimum capacity along P
   - augment flow f by b along P
   - update residual graph
3. return f

Questions

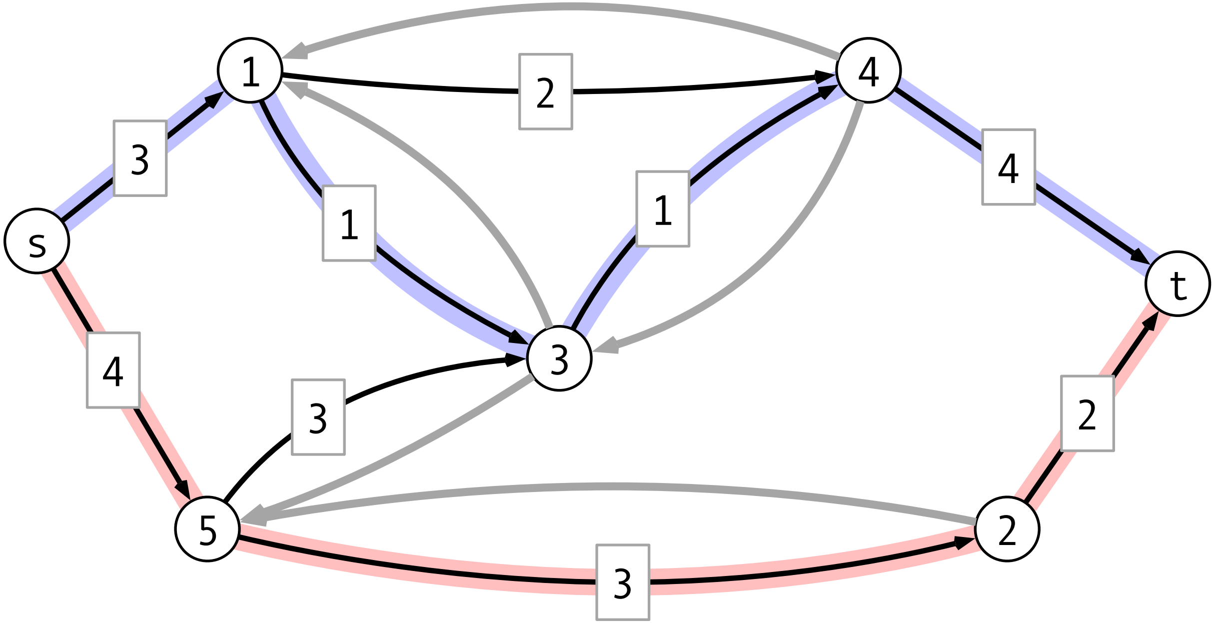
How do we...

1. find a path P from s to t?

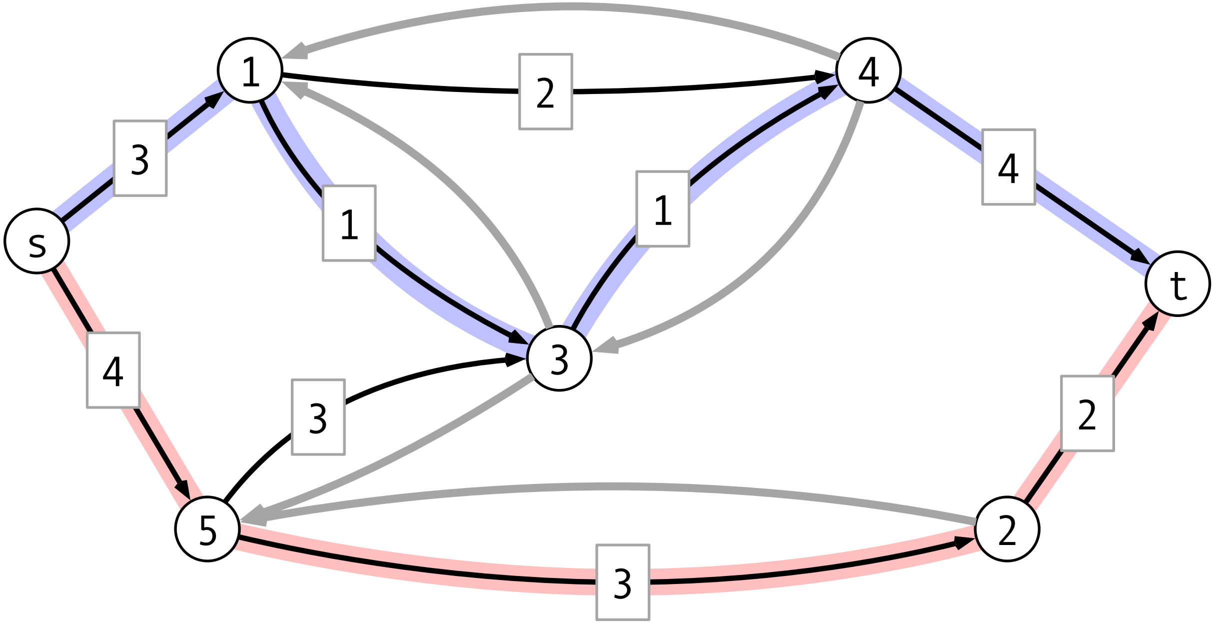
2. update flow f?

3. update residual graph G_f?

Example



e	s,1	s,5	1,3	1,4	2,t	3,4	4,t	5,3	5,2
f(e)	1	2	1		2	1	1		2



e	s,1	s,5	1,3	1,4	2,t	3,4	4,t	5,3	5,2
f(e)	1	2	1		2	1	1		2

Formalizing Ford-Fulkerson

```
MaxFlow(G, s, t):
  Gf <- G
  f <- zero flow
  P <- FindPath(Gf, s, t)
  while P is not null do:
    b <- min capacity of any edge in P
    Augment(Gf, f, P, b)
    P <- FindPath(Gf, s, t)
  endwhile
  return f
```

Augment Procedure

```
Augment(Gf, f, P, b):
  for each edge (u, v) in P
    if (u, v) is forward edge then
      f(u, v) <- f(u, v) + b
      c(u, v) <- c(u, v) - b
      c(v, u) <- c(v, u) + b
    else
      f(v, u) <- f(v, u) - b
      c(v, u) <- c(v, u) + b
      c(u, v) <- c(u, v) - b
```

Running Time

Assume:

1. all capacities are integers
2. $C =$ sum of capacites of edges out of s

Observe:

1. How long to find augmenting path P?

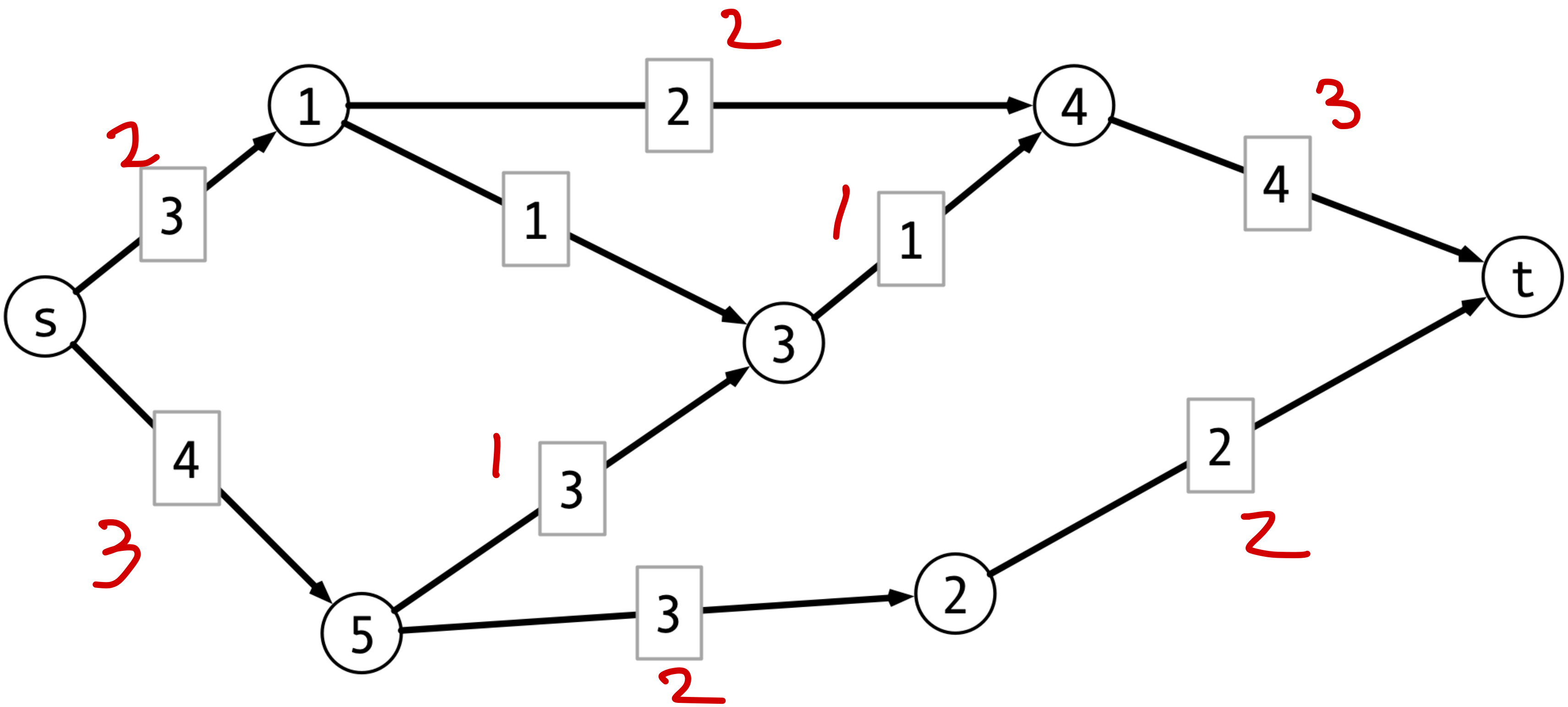
2. How long to run `Augment`?

3. How many iteraions of find/augment?

Conclude: Overall running time?

Optimality of Flow?

Question. How do we know this flow is optimal?



e	s,1	s,5	1,3	1,4	2,t	3,4	4,t	5,3	5,2
f(e)	2	3		2	2	1	~~2~~ 3	1	2

Next Time

Ford-Fulkerson Correctness:

- Maximum Flow = Minimum Cut