Lecture 28: Network Flow

COSC 311 *Algorithms*, Fall 2022

Last Time

directed, weighted graph
u negative edge weights allowed, not neg. cycles

Bellman-Ford Algorithm for SSSP

Definition. For each $j = 0, 1, \dots, n-1$ define $d_j(u, v) =$ length of shortest path from u to v consisting of $\leq j$ hops.

Last Time

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Observations.

1. If G has no negative weight cycles then $d(u, v) = d_{n-1}(u, v)$
2. For all j, $d_j(u, x) = \min(d_{j-1}(u, x), \min_{v \to x} d_{j-1}(u, v) + w(v, x))$

Last Time

Bellman-Ford Algorithm for SSSP

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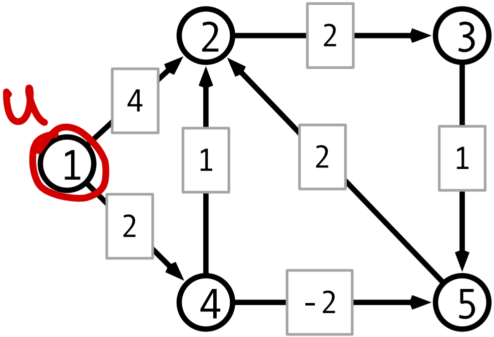
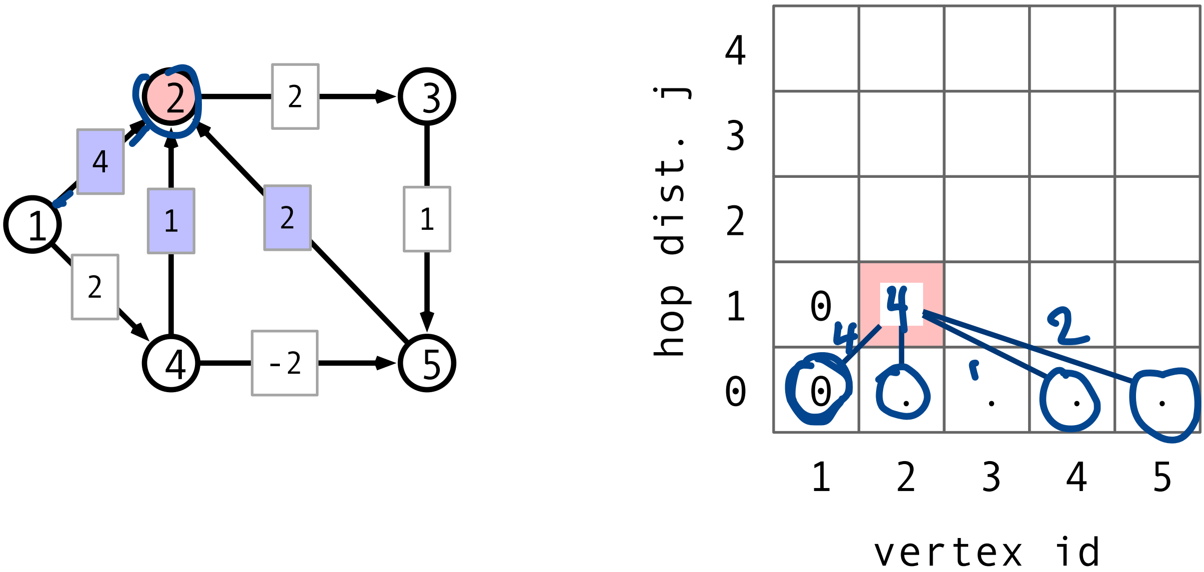
Idea. Use second observation to compute $\underbrace{d_0(u, x)}$, $d_1(u, x), \ldots, d_{n-1}(u, x)$ for all x.



hop dist. j					
4					
3					
2					
1					
0	0	.	.	.	.
	1	2	3	4	5

vertex id



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hop dist. j
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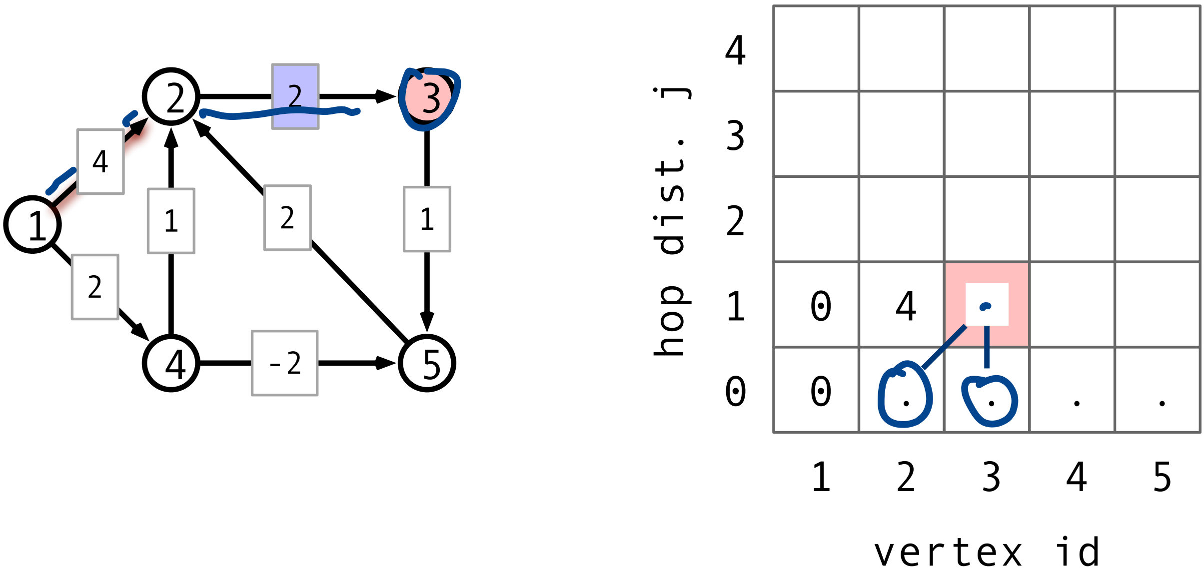
vertex id



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hop dist. j
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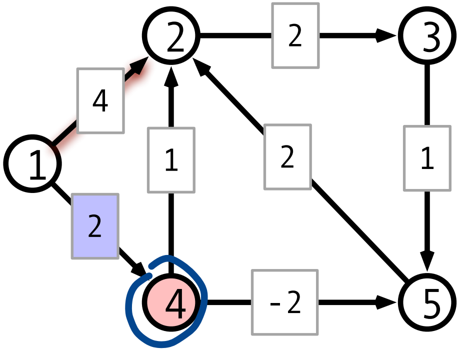
vertex id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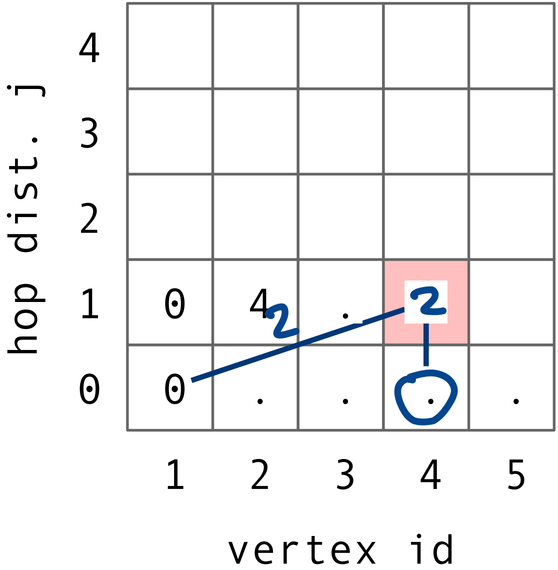
hop dist. j
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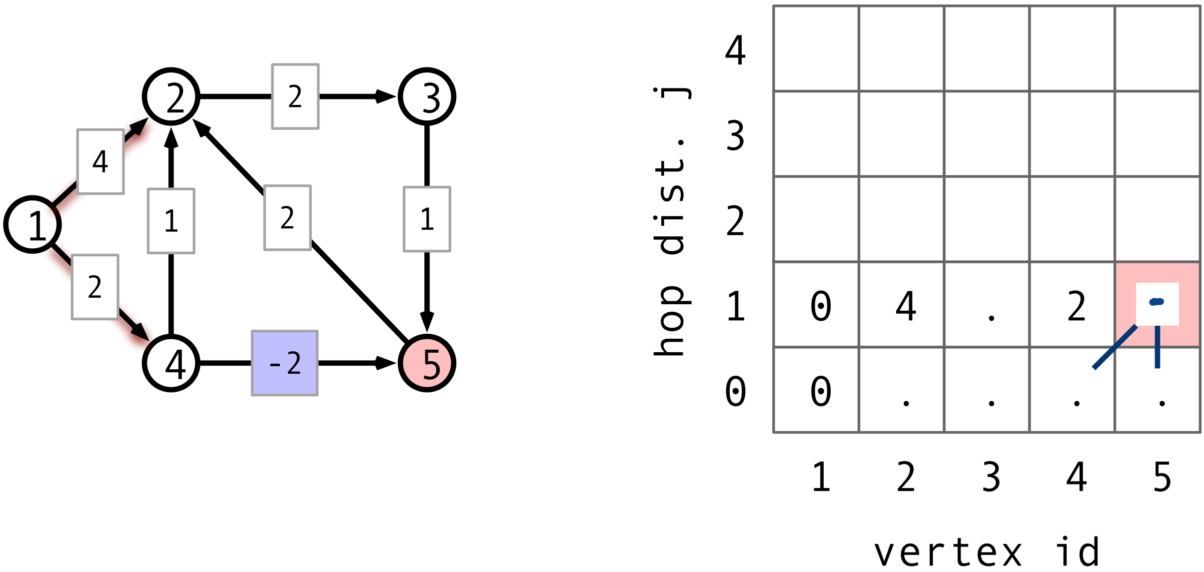
0 . . . .
1 2 3 4 5
vertex id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hop dist. j
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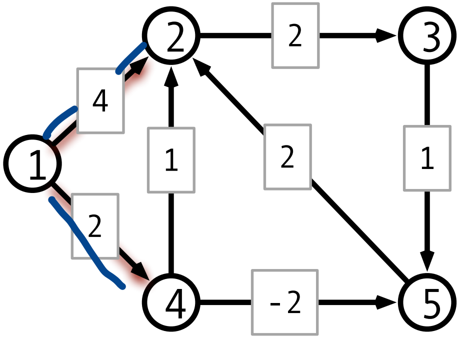
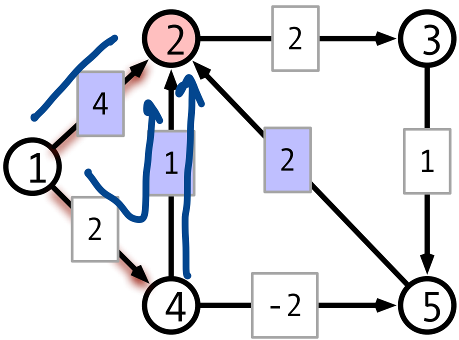
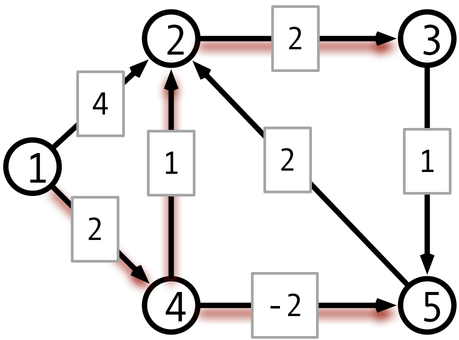


hop dist. j					
4					
3					
2					
1	0	4	.	2	.
0	0	.	.	.	.
	1	2	3	4	5
	vertex id				



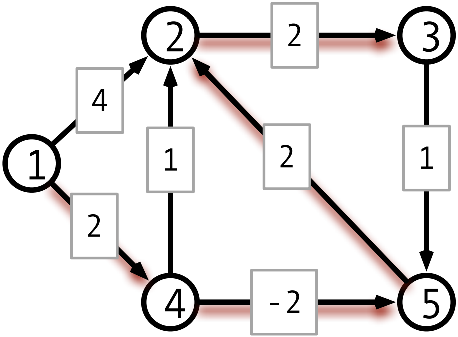
hop dist. j					
4					
3					
2	0	3		2	
1	0	4	.	2	.
0	0	.	.	.	.
	1	2	3	4	5

vertex id



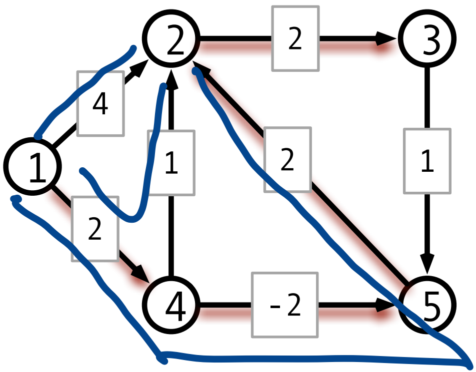
hop dist. j					
4					
3					
2	0	3	6	2	0
1	0	4	.	2	.
0	0	.	.	.	.
	1	2	3	4	5

vertex id



hop dist. j	1	2	3	4	5
4					
3	0	2	5	2	0
2	0	3	6	2	0
1	0	4	.	2	.
0	0	.	.	.	.

vertex id



hop dist. j					
4	0	2	4	2	0
3	0	2	5	2	0
2	0	3	6	2	0
1	0	4	.	2	.
0	0	.	.	.	.
	1	2	3	4	5

vertex id

Bellman-Ford Algorithm

```
Bellman-Ford(V, E, w, u)
  d <- 2d array [0..n-1, 1..n]
  for v = 1 to n do d[0, v] <- infinity
  d[0, u] <- 0
  for j = 1 to n-1 do
    for each vertex v in V set d[j, v] <- d[j-1,v]
    for each vertex v in V
      for each neighbor x of v
        d[j, x] <- Min(d[j, x], d[j-1, v] + w[v, x])
  return d[n-1]
```

n-1

O(m)

Running time is $O(mn)$ if G has n vertices and m edges.

O(mn)

Correctness

val in array

Claim. For all $j = 0, 1, \dots, n-1$ and for all vertices v, $d[j, v]$ stores length of shortest path from u to v with j or fewer hops. I.e., $\boxed{d[j, v]} = \boxed{d_j(v)}$

length of shortest path from u to v w/ at most j hops.

Proof. Induction on j.

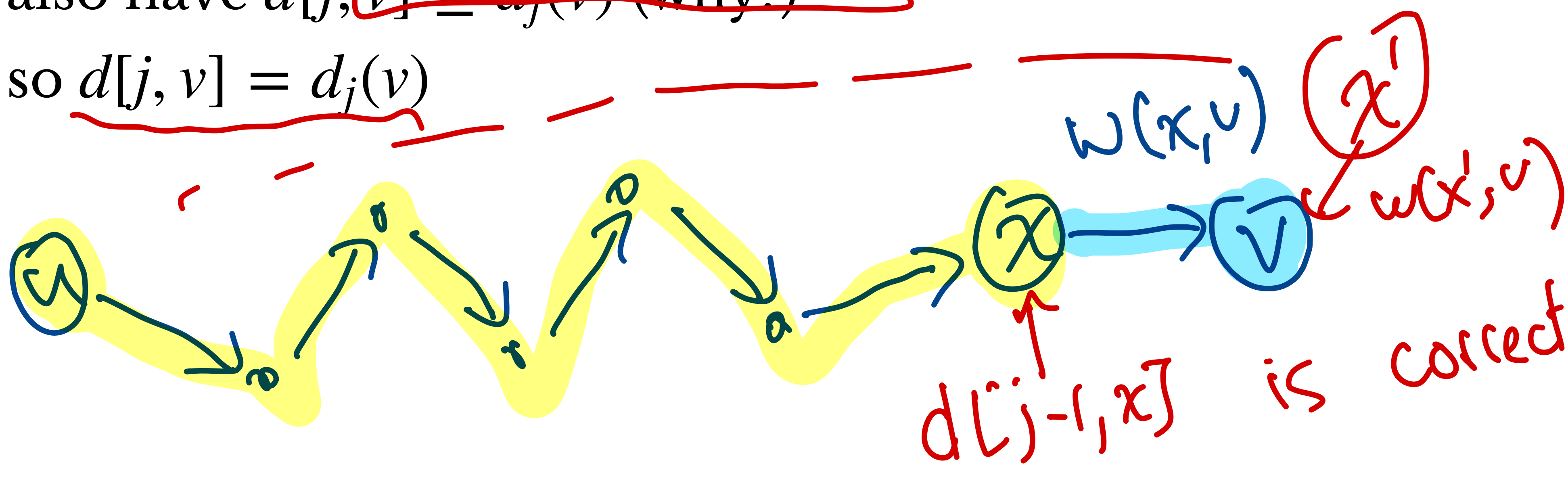
Base case, $j = 0$.

$$d[0,v] = \begin{cases} 0 & u = v \\ \infty & \text{otherwise} \end{cases}$$

✓

Inductive Step, $j - 1 \implies j$

- suppose $d[j-1, v] = d_{j-1}(v)$ for all v
- consider shortest path P of $\leq j$ hops from u to v
- let x be penultimate vertex in P
- then $d_j(v) = d_{j-1}(x) + w(x, v)$
- by inductive hypothesis, $d_{j-1}(x) = d[j-1, x]$
- therefore in iteration j, get
  $d[j, v] \leq d[j-1, x] + w(x, v) = d_{j-1}(x) + w(x, v) = \boxed{d_j(v)}$
- also have $d[j, v] \geq d_j(v)$ (why?)
- so $d[j, v] = d_j(v)$



Conclusion

If G has no negative weight cycles, then Bellman-Ford solves single source shortest paths in $O(mn)$ time.

Dijkstra vs Bellman-Ford?

Running times:

- Dijkstra: $O(m \boxed{\log n})$
- Bellman-Ford: $O(m\boxed{n})$

$\log n << n$

Why pick Bellman-Ford over Dijkstra?

if G has negative weights!

Dijkstra vs Bellman-Ford?

Running times:

- Dijkstra: $O(m \log n)$
- Bellman-Ford: $O(mn)$

Why pick Bellman-Ford over Dijkstra?

- Why might Bellman-Ford be preferable even if graph has no negative weight edges?

Bellman-Ford Again

```
Bellman-Ford(V, E, w, u)
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  for v = 1 to n do d[0, v] <- infinity
  d[0, u] <- 0
  for j = 1 to n-1 do
    for each vertex v in V set d[j, v] <- d[j-1,v]
    for each vertex v in V
      for each neighbor x of v
        d[j, x] <- Min(d[j, x], d[j-1, v] + w[v, x])
  return d[n-1]
```
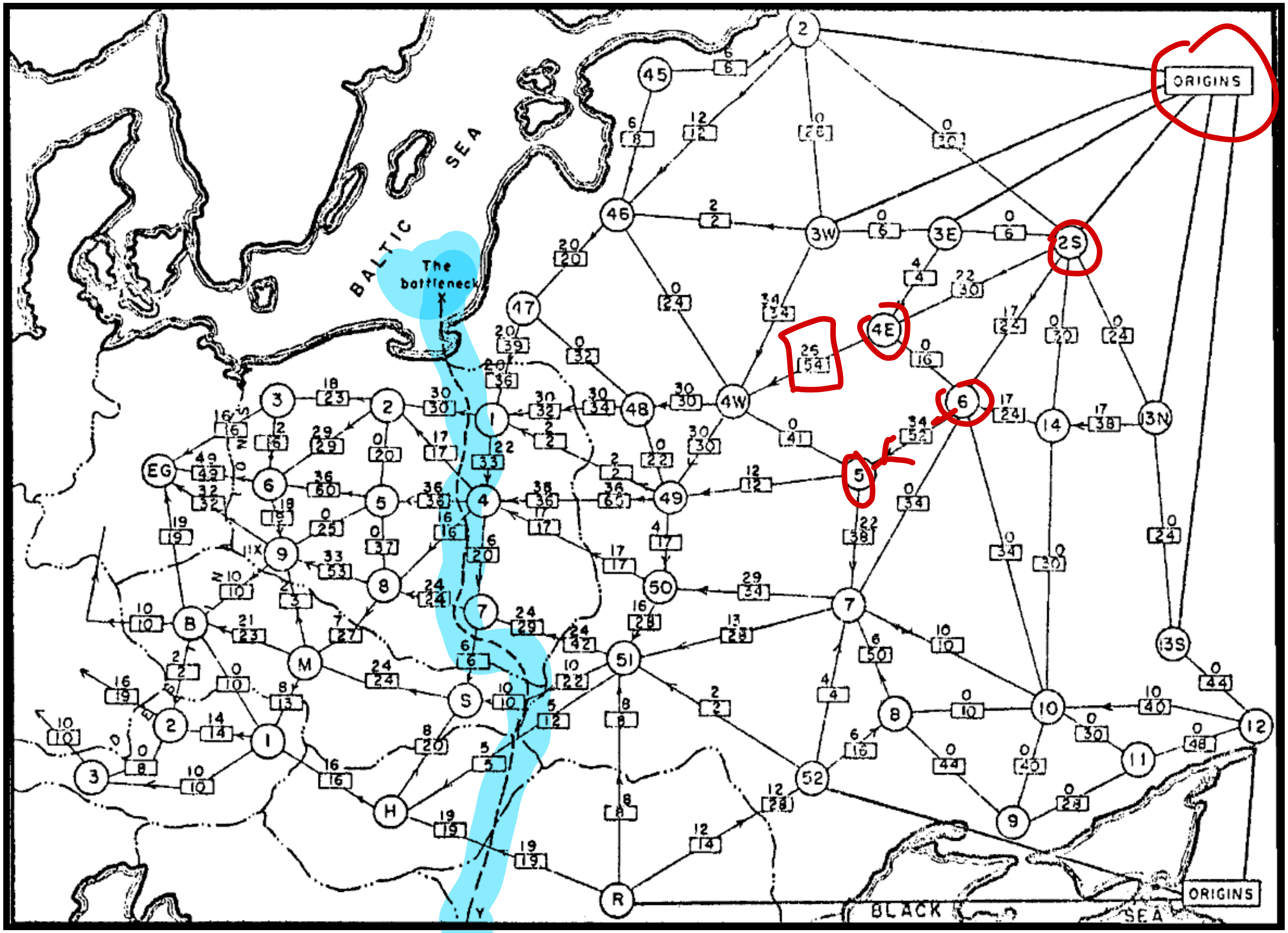

"Distributed Algorithm"

Cold War

Soviet Rail Network, ca. 1955



Networks to Graphs

Modeling the network:

- nodes represent railway junctions
- edges represent rail lines
- weights represent capacities of lines
  - capacity = tonnage that can cross line per unit time
  - proportional to cost of disrupting line

Networks to Graphs

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Question 1. How much material can the USSR transport to Western Europe per unit time?

Question 2. What is the cheapest way to disrupt flow of all material?

- Harris & Ross, 1955 USAF, declassified 1999

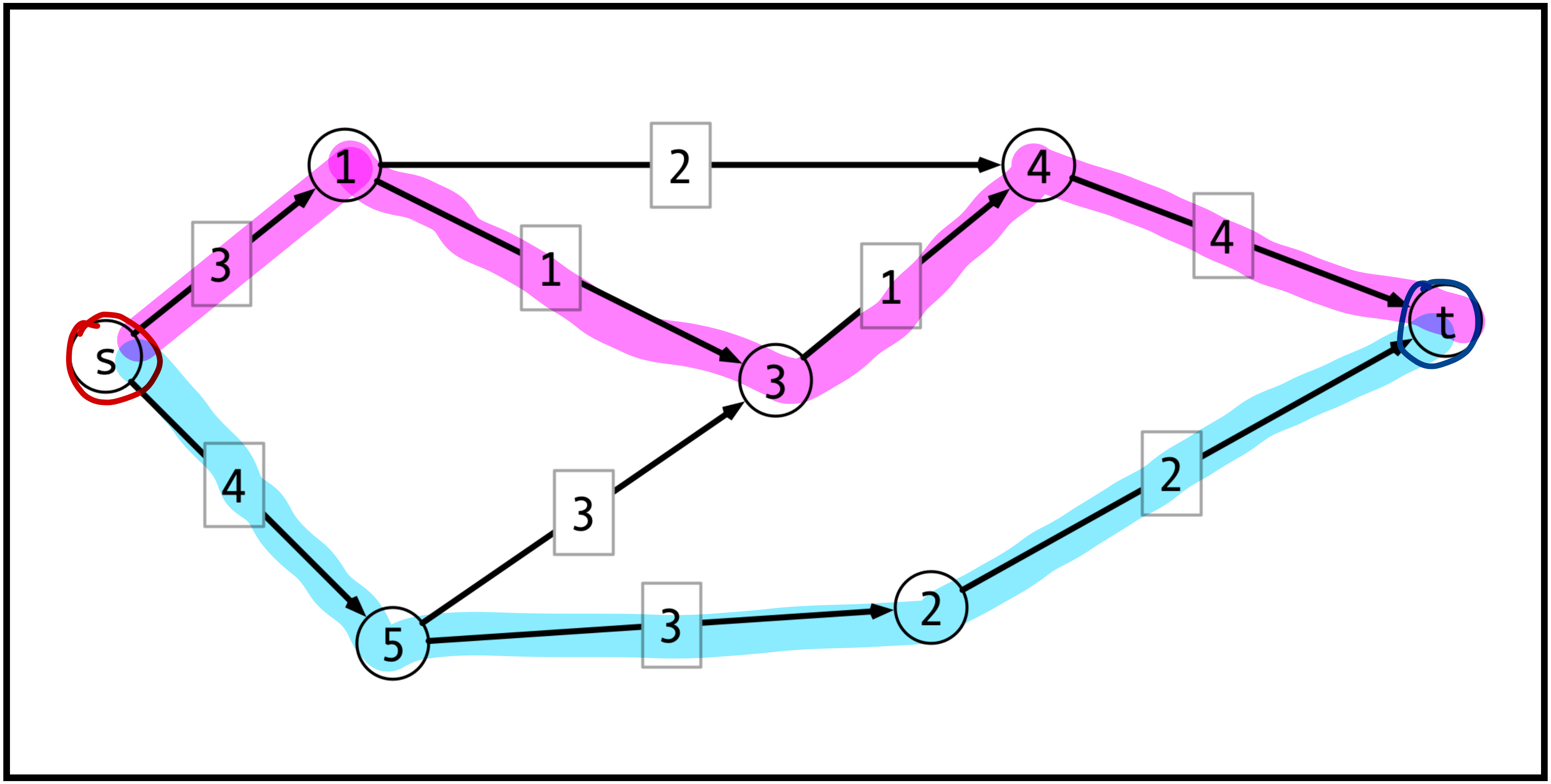
Network Flow

A new interpretation of directed graphs:

- network of (directional) pipes
- weights are *capacities*
  - how much fluid can flow through pipe~~r~~ per time
- designated *source node s*
  - all edges directed away from *s*
- designated *sink* or *destination node t*
  - all edges directed towards *t*

Question. How much fluid be routed from *s* to *t* per unit time?

Example



Flows, Formally

Setup.

- $G = (V, E)$ a directed graph, s, t source and sink
- $c(u, v)$ is capacity of edge (u, v)

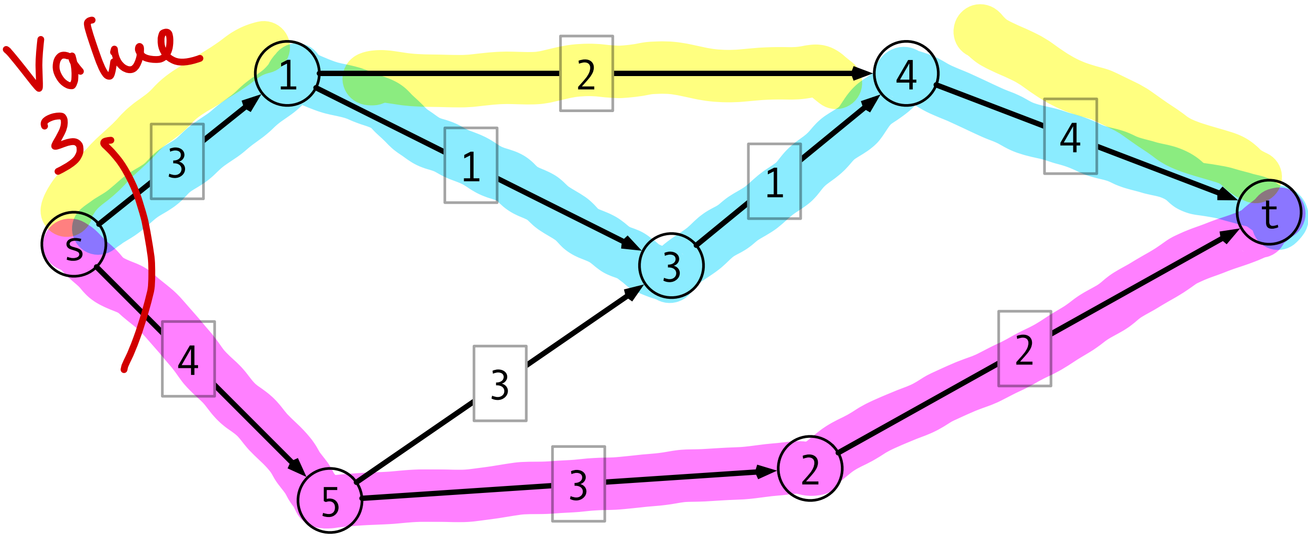
f(e) = amount of flow crossing e

Flows. An **s-t flow** f is a function $f : E \to \mathbf{R}^+$ satisfying:

1. *capacity constraints:* for each edge e, $f(e) \leq c(e)$
2. *conservation:* for every vertex $v \neq s, t$, flow into v = flow out of v:
   - $\sum_{x \to v} f(x, v) = \sum_{v \to y} f(v, y)$

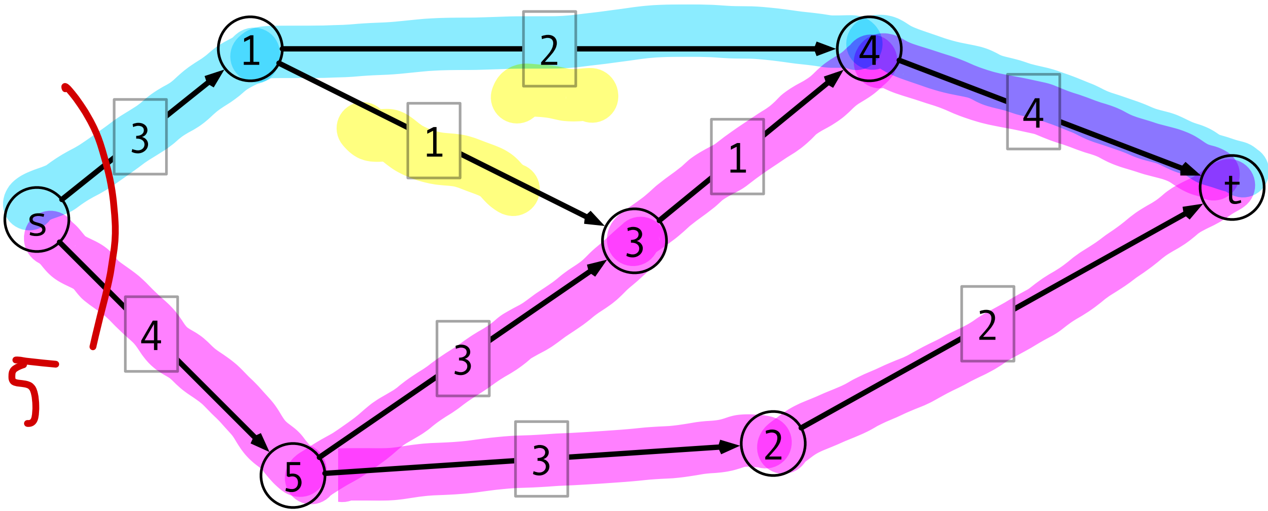
The **value** of the flow f is $\mathrm{val}(f) = \sum_{s \to v} f(s, v)$

Flow Example 1



e	s,1	s,5	1,3	1,4	2,t	3,4	4,t	5,3	5,2
f(e)	1	2	1		2	1	1		2

Flow Example 2



e	s,1	s,5	1,3	1,4	2,t	3,4	4,t	5,3	5,2
f(e)	2	3		2	2	1	3	1	2

Max Flow Problem

Input.

- weighted directed graph $G = (V, E)$
  - weights = edge capacities > 0
- source s, sink t
  - all edges oriented out of s
  - all edges oriented into t

Output.

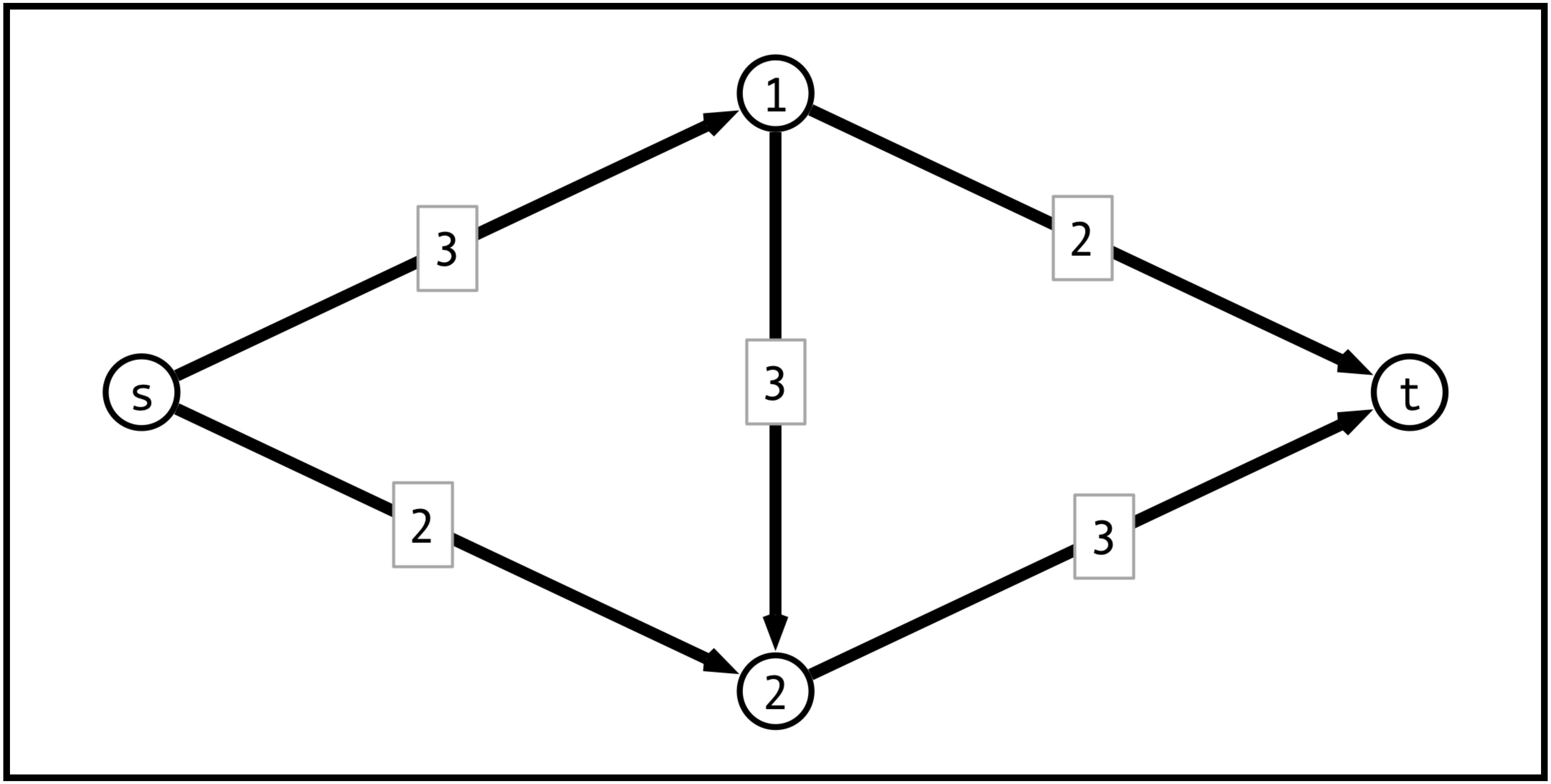
- flow f of maximum value
  - val(f) $= \sum_{s \to v} f(s, v)$

A Simple Greedy Strategy

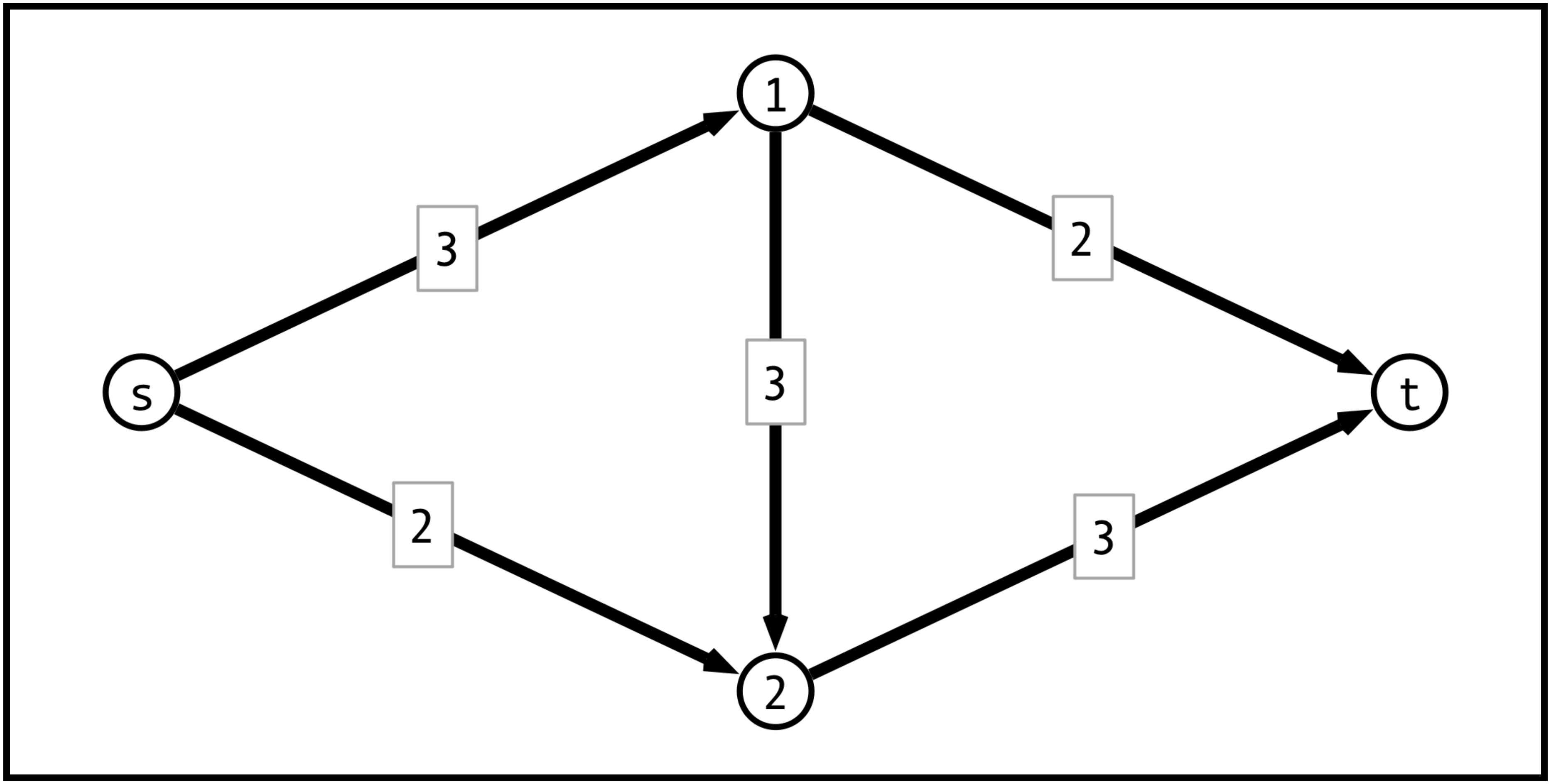
Repeat until done:

1. find an “unsaturated” path P from s to t
2. find minimum (remaining) capacity b along P
3. route b units of flow along P

Greedy Approach Example



Choosing Different First Path



Greedy Issue

Flow along P may block other viable paths

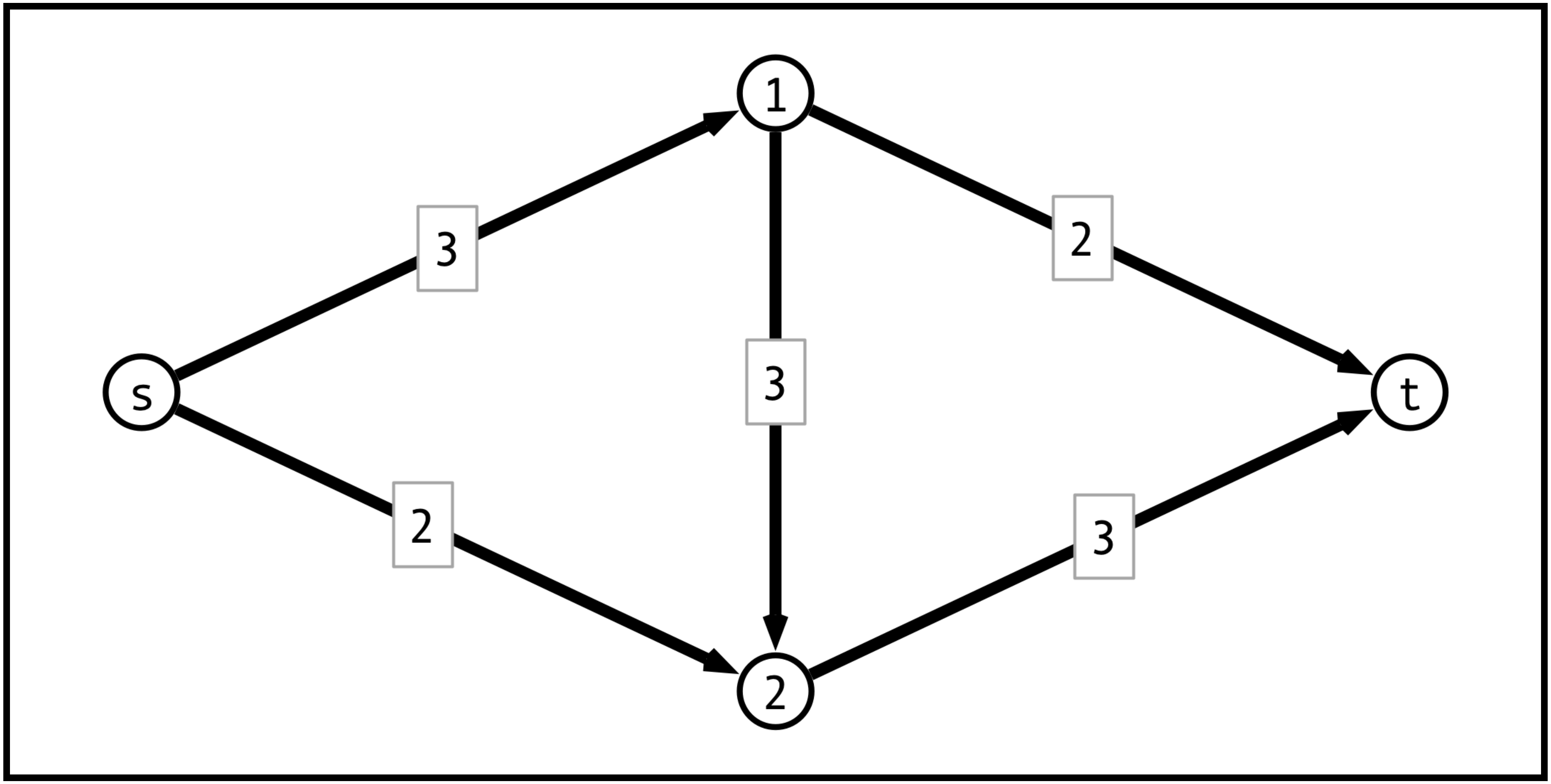
Question. How to fix this?

Augmenting Paths

Idea. Add “undo” feature for each edge

- if f routes $f(u, v) \leq c(u, v)$ flow from u to v, add reverse edge (v, u) with capacity $c(v, u) = f(u, v)$
- using (v, u) corresponds to “pushing back” flow from (u, v)
- if an alternate route for this flow can be found, then more flow can be routed through u

Pushing Back Example



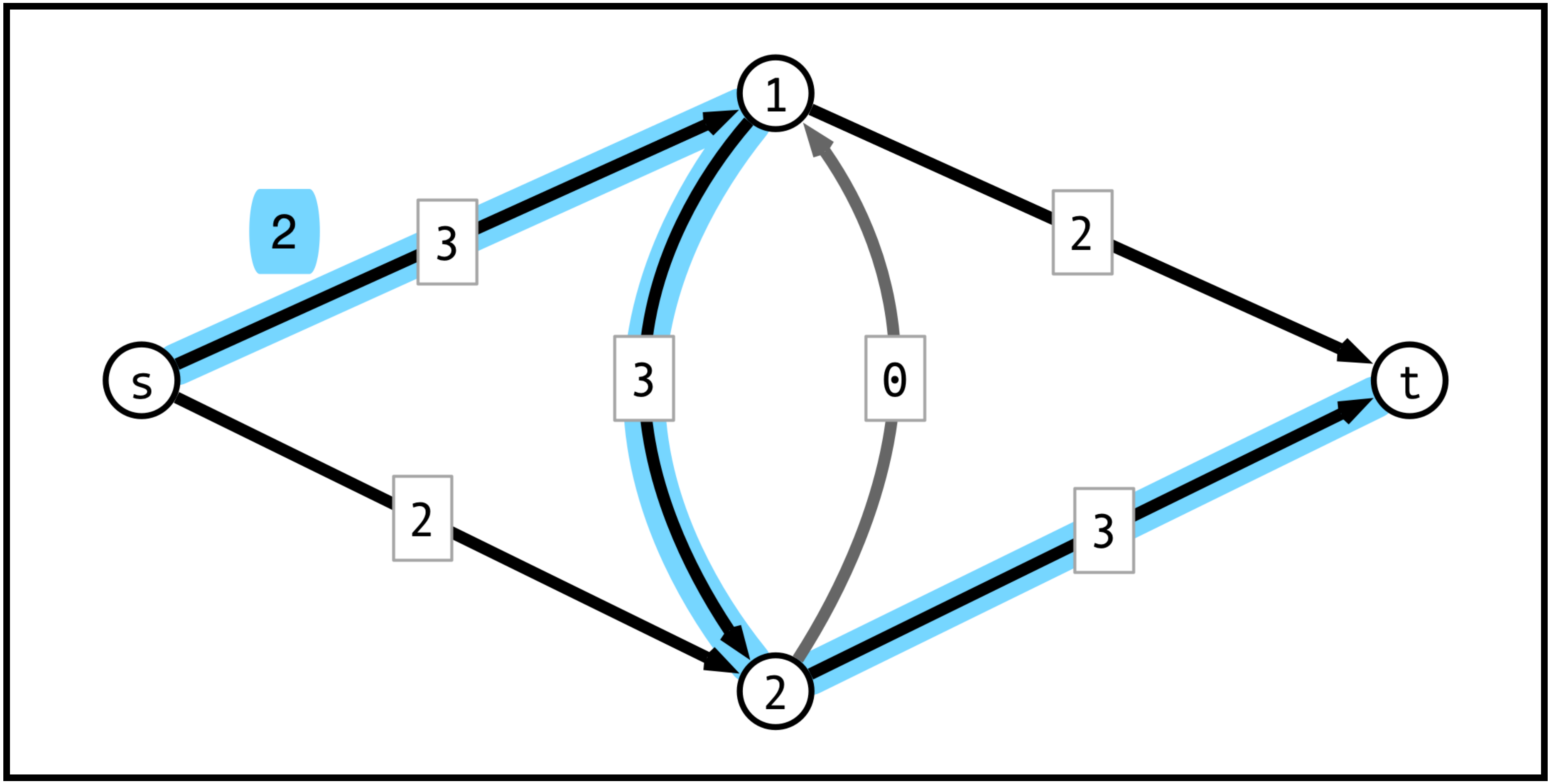
The Residual Graph

- $G = (V, E)$ original graph
- f a flow on G

Residual graph $G_f = (V_f, E_f)$

- vertex set $V_f = V$
- for each $(u, v) \in E$, add (v, u) to E_f
  - (u, v) is *forward edge*
  - (v, u) is *backward edge*
- in G_f capacity of (u, v) is:
  - $c(u, v) - f(u, v)$ if $(u, v) \in E$ (forward edge)
  - $f(v, u)$ if $(v, u) \in E$ (backward edge)

Residual Graph Example



Ford-Fulkerson Algorithm

Very high level

1. Initialize residual graph, flow f
2. While there is a path from s to t in residual graph do:
    - find path P from s to t
        - ignore edges with capacity 0
    - $b \leftarrow$ minimum capacity along P
    - augment flow f by b along P
    - update residual graph
3. return f

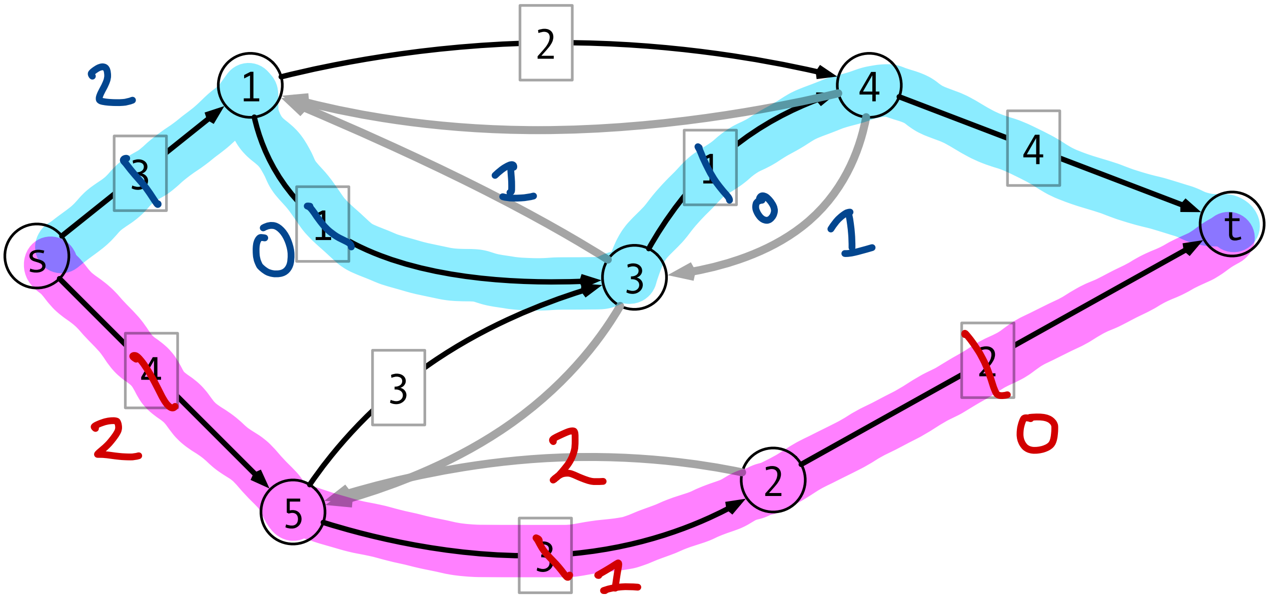
Question

We've found a path P with minimum capacity $b > 0$!

Question. How do we...

1. update flow f?
2. update residual graph G_f?

Example



e	s,1	s,5	1,3	1,4	2,t	3,4	4,t	5,3	5,2
f(e)	1	2	1		2	1	1		2

Formalizing Ford-Fulkerson

```
MaxFlow(G, s, t):
  Gf <- G
  f <- zero flow
  P <- FindPath(Gf, s, t)
  while P is not null do:
    b <- min capacity of any edge in P
    Augment(Gf, f, P, b)
    P <- FindPath(Gf, s, t)
  endwhile
  return f
```

Augment Procedure

```
Augment(Gf, f, P, b):
  for each edge (u, v) in P
    if (u, v) is forward edge then
      f(u, v) <- f(u, v) + b
      c(u, v) <- c(u, v) - b
      c(v, u) <- c(v, u) + b
    else
      f(v, u) <- f(v, u) - b
      c(v, u) <- c(v, u) + b
      c(u, v) <- c(u, v) - b
```

Running Time

Assume:

1. all capacities are integers
2. $C =$ sum of capacites of edges out of s

Observe:

1. How long to find augmenting path P?

2. How long to run `Augment`?

3. How many iteraions of find/augment?

Conclude: Overall running time?

Next Time

Ford-Fulkerson Correctness!