Lecture 22: Intro to Dynamic Programming

COSC 311 *Algorithms*, Fall 2022

Announcements

1. ‘22E Honors Thesis talks Today!
   - 4-5pm in SCCE A131

2. Courses Next Semester
   - 225 Algorithms and Visualization
   - 273 Parallel and Distributed Computing

Overview

1. Memoization
2. Profit Maximization, Revisited

So Far

Algorithmic Paradigms

1. Divide and Conquer

2. Greedy

And now...

Dynamic Programming

Features of Dynamic Programming

1. Break problem into smaller sub-problems
2. *Iterate* over sub-problems to produce solution

Compare to divide and conquer:

1. Break problem into smaller sub-problems
2. *Recursively* solve sub-problems
3. Combine solutions

Issues with Recursion

Recall the **Fibonacci Sequence:**

- 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

Issues with Recursion

Recall the **Fibonacci Sequence:**

- $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

Defined by:

base cases

1. $f(1) = f(2) = 1$

recursion relation

2. for $n > 2, f(n) = f(n-1) + f(n-2)$

Issues with Recursion

Recall the **Fibonacci Sequence:**

- 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

Defined by:

1. $f(1) = f(2) = 1$
2. for $n > 2, f(n) = f(n-1) + f(n-2)$

Recursive code:

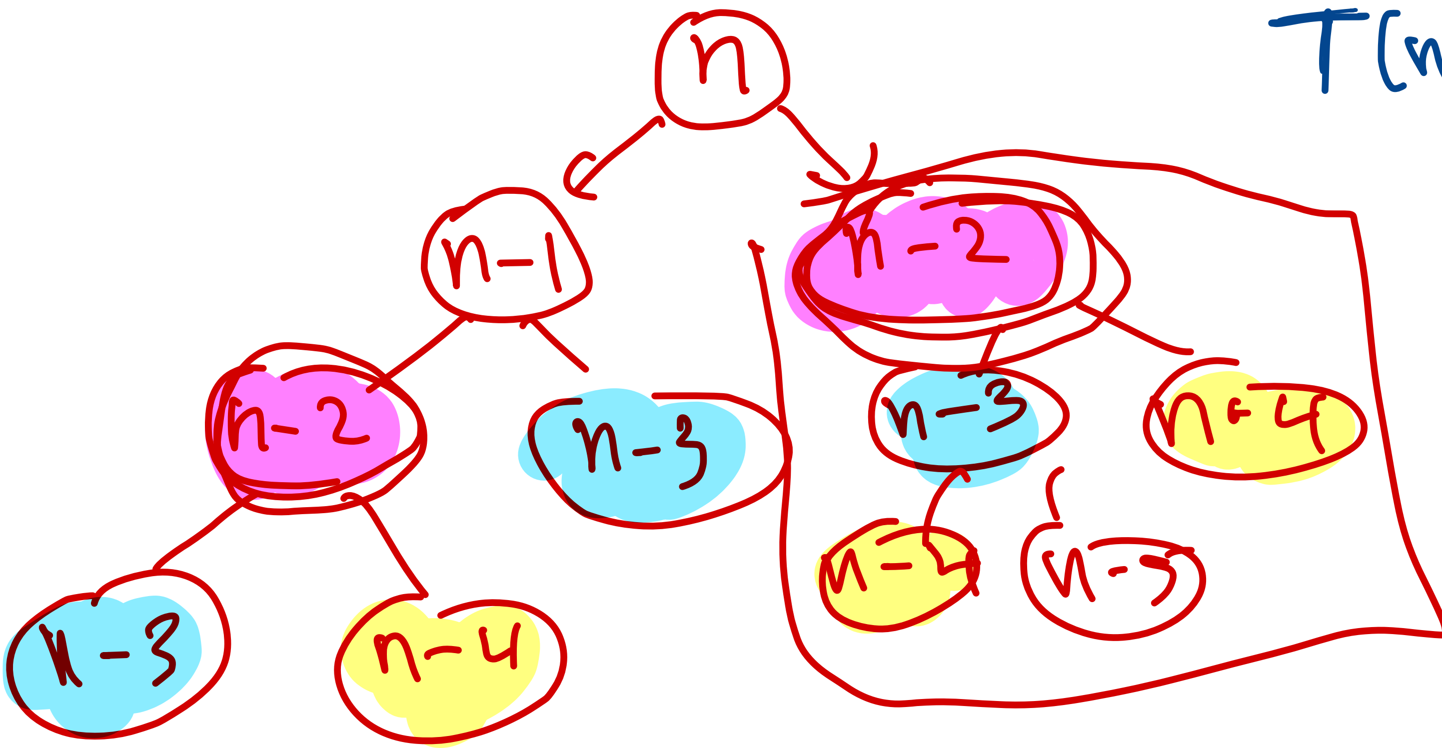
```
Fib(n):
  if n <= 2 then return 1      — base
  return Fib(n-1) + Fib(n-2)   — recursive calls.
```

What is Running Time?

```
Fib(n):
   if n <= 2 then return 1
   return Fib(n-1) + Fib(n-2)
```

$\Theta(1)$

$\Theta(1)$



$$T(n) = T(n-1) + T(n-2) + \Theta(1)$$

$$\geq 2 \cdot T(n-2) + \Theta(1)$$

$$\geq 4 \cdot T(n-4) +$$

$$\geq 8 \cdot T(n-6)$$

$$\vdots$$

$$\geq 2^k T(n-2k)$$

$$k = \frac{n-1}{2} \implies T(n) \geq 2^{\frac{n-1}{2}} \cdot T(1) = \Theta\left(2^{\frac{n-1}{2}}\right)$$

Redundant Recursive Calls

Recursive subproblems *overlap*

Memoization

Recursing without recursion

- store results of recursive calls
- iterate over results rather than making recursive calls
  - compute results "bottom up" rather than "top down"
  - iterate over stored values to compute "next" value

How to do for Fibonacci?

	1	2	3	4	5	6	...
a:	[1	1	2	3	5	8	...]

$a[3] = a[2] + a[1]$

Iterate over $i = 3, 4, 5, \ldots, n$

set $a[i] \leftarrow a[i-1] + a[i-2]$

Memoized Fibonacci

```
MFib(n):
  a <- array of size n
  a[1] <- 1
  a[2] <- 1
  for each index i from 3 to n do
    a[i] <- a[i-1] + a[i-2]
  endfor
  return a[n]
```

n - 2 iterations

$O(n)$.

Running Time?

```
MFib(n):
  a <- array of size n
  a[1] <- 1
  a[2] <- 1
  for each index i from 3 to n do
    a[i] <- a[i-1] + a[i-2]
  endfor
  return a[n]
```

The Moral

With memoization, we converted

- recursive procedure
- many redundant recursive calls
- running time $\Omega(2^{n/2})$

into

- iterative procedure
- running time $O(n)$

Profit Maximization

Recall: Profit Maximization

Goal. Pick day b to buy and day s to sell to maximize profit.

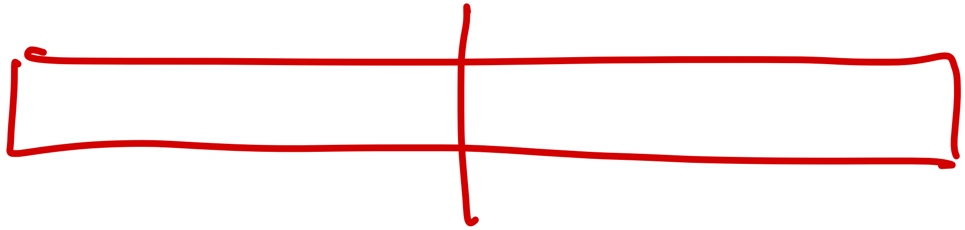
Formalizing the Problem

Input. Array a of size n

- $a[i]$ = price of Alphabet stock on day i

Output. Indices b (buy) and s (sell) with $1 \leq b \leq s \leq n$ that maximize profit

- $p = a[s] - a[b]$



Divide and Conquer Algorithm

```
MaxProfit(a, i, j):
  if j - i = 1 then return 0
  m <- (i + j) / 2
  left <- MaxProfit(a, i, m)
  right <- MaxProfit(a, m, j)
  min <- FindMin(a, i, m)
  max <- FindMax(a, m, j)
  return Max(left, right, max - min)
```

$\Theta(n)$ time

Running time?

Exercise: $O(n \log n)$ from M.T.

Another $\boxed{\text{(Recursive}})$ Procedure?

- consider last day, n
- two cases for optimal solution:
  1. max profit achieved by selling on day n
  2. max profit achieved by selling before day n

Questions.

1. In case 1, how should we determine buy date?

   find min value in a[1..n], buy on that day

2. In case 2, how should we compute max profit?

   Recursion: Solve on a[1..n-1].

Recursive Procedure

```
MaxProfit(a, n):
  if n = 1 then return 0
  min <- FindMin(a, n)
  max <- MaxProfit(a, n-1)
  return max(a[n] - min, max)
```

$\Theta(n)$

selling before
day n profit

Running time?

Profit selling on day n,
buying @ min price

n — $\Theta(n)$

$n-1$ — $\Theta(n-1)$

$n-2$ — $\Theta(n-2)$

$n-3$

$\vdots$

1 … $\Theta(1)$

add up $\rightarrow \Theta(n^2)$

Questions

1. What makes `MaxProfit` inefficient?

   Duplicated work
   computing min value

2. What part(s) of the procedure can be memoized?

   this

Memoizing `MaxProfit`

Create two arrays:

1. `min[i]` stores minimum value in `a[1..i]`
2. `max[i]` stores maximum profit achievable by selling up to time `i`

Question. How to update these arrays?