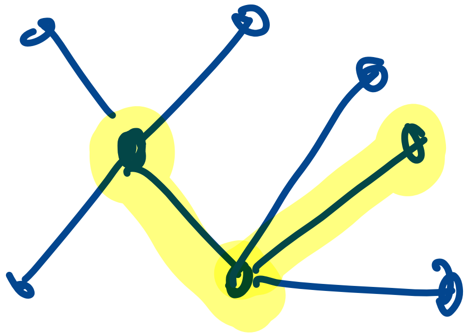
Lecture ~~19~~ 20: Minimum Spanning Trees, Part 3

COSC 311 *Algorithms*, Fall 2022

Announcements

1. Masks still required in class
2. No class on Monday 10/24 ←
3. HW 03, Question 1:
    - $n = 2^B$. B = # of bits
    - array contains $0, 1, \ldots, n-1$ (not in order)
    - values represented as B bit numbers
4. HW 03 now due Sunday

Last Time



Prim's algorithm for Minimum Spanning Trees:

- Grow a tree from an arbitrary seed vertex
- Each step, add minimum weight edge out of tree

Cut Claim:

- if T an MST, $[U, V - U]$ a cut, e min weight cut edge
- then T contains e

Prim correctness follows from cut claim

MSTs, Another Way

Prim:

- Grow tree greedily from a single seed vertex
- Maintain a (connected) tree

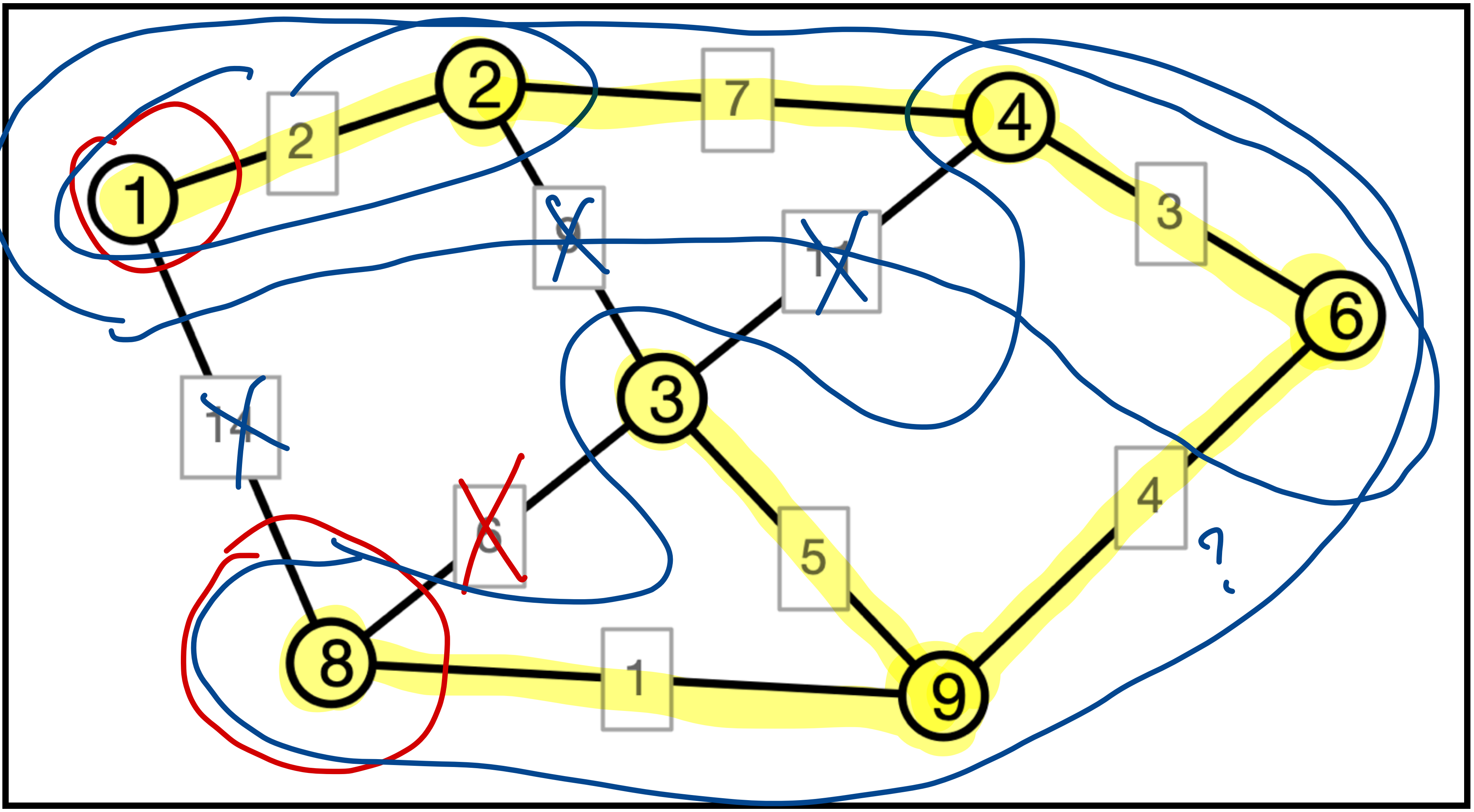
Edge Centric View:

- Maintain a collection of edges (not necessarily a tree)
- Add edges to collection to eventually build an MST

Questions:

- How to *prioritize* edges?
- How to determine whether or not to include an edge?

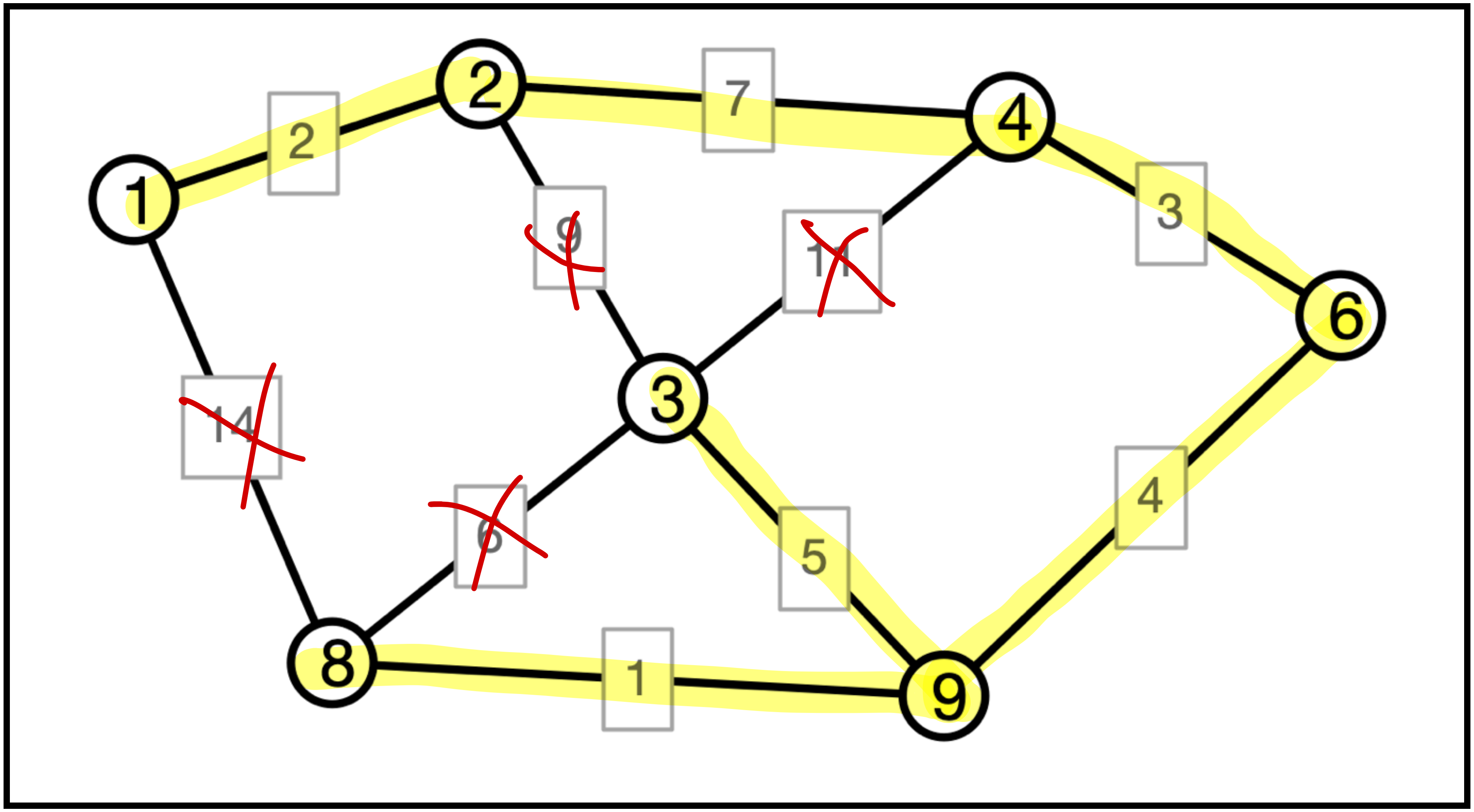
Picture



Kruskal’s Algorithm

```
Kruskal(V, E, w):
  C <- collection of components
    initially, each vertex is own component
  F <- empty collection
  # iterate in order of increasing weight
  for each edge e = (u, v) in E
    if u and v are in different components then
      add (u, v) to F
      merge components containing u and v
    endif
  endfor
  return F
```

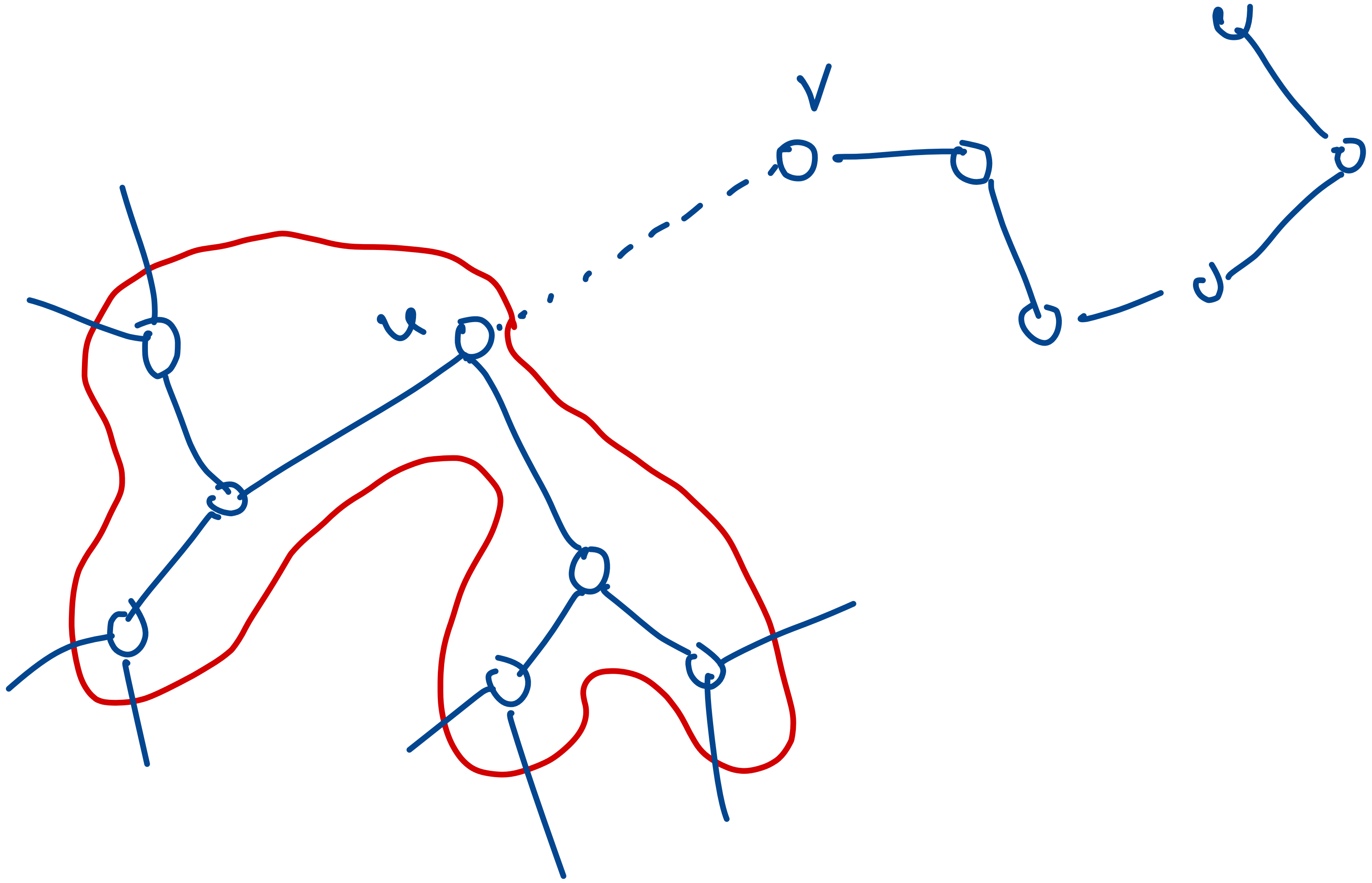
Kruskal Illustration



Kruskal Correctness I

Claim 1. Every edge added by Kruskal must be in every MST.

Why?



Kruskal Correctness I

Claim 1. Every edge added by Kruskal must be in every MST.

Why?

- Suppose $e = (u, v)$ added by Kruskal
- Consider the cut $U, V - U$ where U is u's component
- e is lightest edge across the cut (why?)
- therefore e must be in MST (why?)

$\hookrightarrow$ By Cut claim

Kruskal Correctness II

Claim 2. Kruskal produces a spanning tree.

Why?

Kruskal Correctness II

Claim 2. Kruskal produces a spanning tree.

Why?

- edges added by Kruskal do not contain cycles (why?)

  Never add edge that creates a cycle

- edges added by Kruskal connect graph (why?)

  Only don't include an edge if endpts are already connected

Conclusion

Theorem. Kruskal's algorithm produces an MST.

Next Question. How could we implement Kruskal's algorithm efficiently? What is its running time?

Kruskal's Algorithm

```
Kruskal(V, E, w):
  C <- collection of components
    initially, each vertex is own component
  F <- empty collection
  # iterate in order of increasing weight
  for each edge e = (u, v) in E
    if u and v are in different components then
      add (u, v) to F
      merge components containing u and v
    endif
  endfor
  return F
```

sorting: $O(m \log n)$

$O(1)$

$O(n)$

Costly Operations

1. Get edges in order of increasing weight
2. Determine if *u* and *v* are in same component
3. Merge two components

Costly Operations

1. Get edges in order of increasing weight
2. Determine if u and v are in same component
3. Merge two components

Question. How to get edges in order of increasing weight?

- e.g. Use BFS to find all edges, add to priority queue
  - ↳ repeatedly remove min

→ Alt: add edges to array and sort by weight

$O(m \log n)$

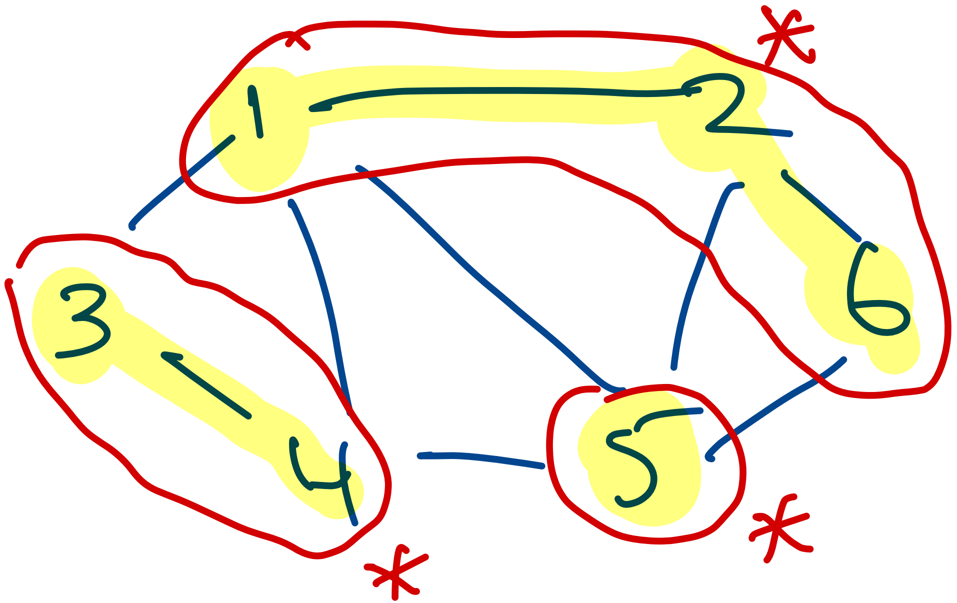
Maintaining Components

Idea. For each component, designate a **leader**

- leader is a vertex in its component
- maintain an array that stores each vertex's component's leader
  - `leader[i] = v` means that v is leader of `i`'s component

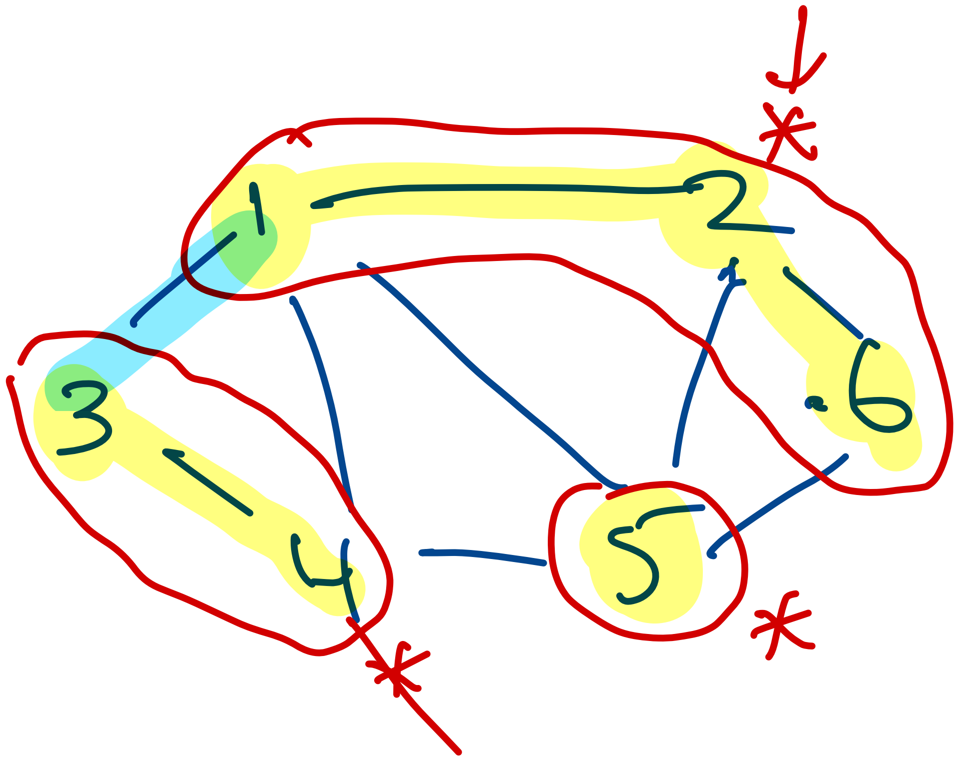
Question. How to check if vertices `i` and `j` are in the same component? Running time?

1	2	3	4	5	6
2	2	4	4	5	2



Merging Components

Question. How to merge two components?



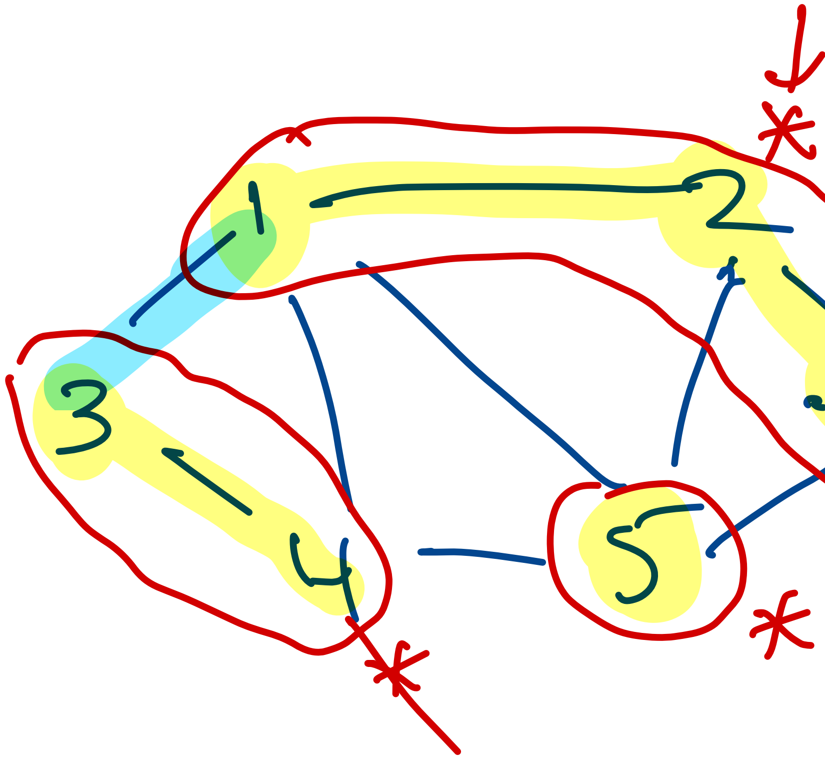
1	2	3	4	5	6
2	2	4	4	5	2

$\Downarrow$

1	2	3	4	5	6
2	2	~~4~~	~~4~~	5	2
		2	2		

Merging Components Efficiently?

For each leader, maintain list of vertices in its component.



1	2	3	4	5	6
2	2	~~4~~	~~4~~	5	2
		2	2		

2 : 1, 2, 6 ← append

4 : 3, 4

5 : 5

Time to merge u w/ v :

O(size of smaller component).

Merging Components Efficiently?

For each leader, maintain list of vertices in its component.

To merge components with leaders *u* and *v*:

1. choose *u* or *v* to be leader of merged component
   - how? larger component
2. if *u* is new leader
   - for each vertex *x* on *v*'s list
     - add *x* to *u*'s list
     - set *x*'s leader to *v*

Question. Running time?

$O($smaller comp size$)$.

Merging Strategy

When merging components with leaders u and v, new leader is leader of larger component

Claim. If x is relabeled k times, then x's component has size at least 2^k.

Consequence

Claim. If x is relabeled k times, then x's component has size at least 2^k.

Consequence 1. If x's component has size ℓ, then x was relabeled at most $\log \ell$ times.

Consequence

Claim. If x is relabeled k times, then x's component has size at least 2^k.

Consequence 1. If x's component has size ℓ, then x was relabeled at most $\log \ell$ times.

Consequence 2. Running time of all merge operations in Kruskal is $O(n \log n)$

Conclusion

Theorem. Kruskal's algorithm can be implemented to run in time $O(m \log n)$ in graphs with n vertices and m edges.

Conclusion

Theorem. Kruskal's algorithm can be implemented to run in time $O(m \log n)$ in graphs with n vertices and m edges.

Remark. More efficient data structures for merging sets exist

- "Union-find" ADT, "disjoint-set forest" data structure
- time to perform merges is $O(n\alpha(n))$
  - $\alpha(n)$ is "inverse Ackerman function"
  - $\alpha(n)$ grows so slowly, it is practically constant

Next Time

- Interval Scheduling (recorded lecture)