Lecture 18: Minimum Spanning Trees

COSC 311 *Algorithms*, Fall 2022

Announcements

1. Midterms
   - should finish grading tomorrow
   - pick up in class Wednesday or OH on Thursday
2. No class on Monday 10/24
   - reading + recorded lecture instead

3. HW 03 Posted, due Friday

4. Masking Survey ←

Overview

1. Minimum Spanning Tree Problem
2. Prim’s Algorithm

Last Time: Dijkstra's Algorithm

Single Source Shortest Paths

- **Input**
  - graph $G = (V, E)$
  - edge weights/lengths w
  - starting vertex u
- **Output**
  - (weighted) distance $d[v] = d_w(u, v)$ for every vertex v

Last Time: Dijkstra's Algorithm

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Dijkstra's Algorithm:

Note: if all weights are 1, Dijkstra is equiv. to BFS

- maintain set S of finalized nodes
- *greedily* choose $v \in V - S$ with minimum $d[v]$
  - finalize v
  - update $d[x]$ for each neighbor x of v

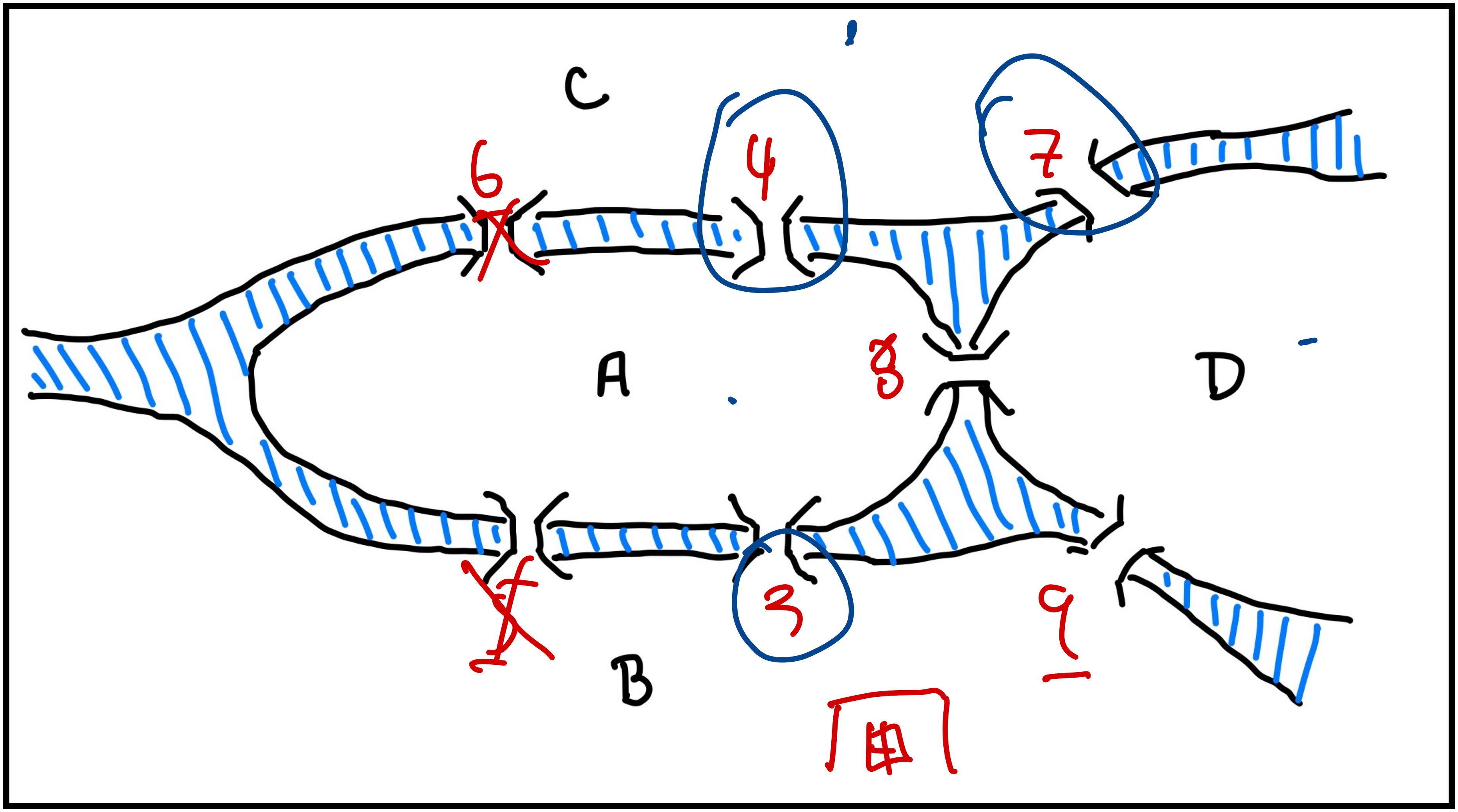
DPW in Königsberg

Winter is coming to Königsberg!

- Must prepare the bridges for ice
- Each bridge has an associated cost to de-ice
- Königsberg doesn't have the budget to de-ice all bridges
- For public safety, must ensure that all landmasses are reachable from each other

Question. How to find the *cheapest* set of bridges to de-ice that maintain connectivity?

Picture



Graph Problem

Input:

- a weighted graph $G = (V, E)$ with edge weights w

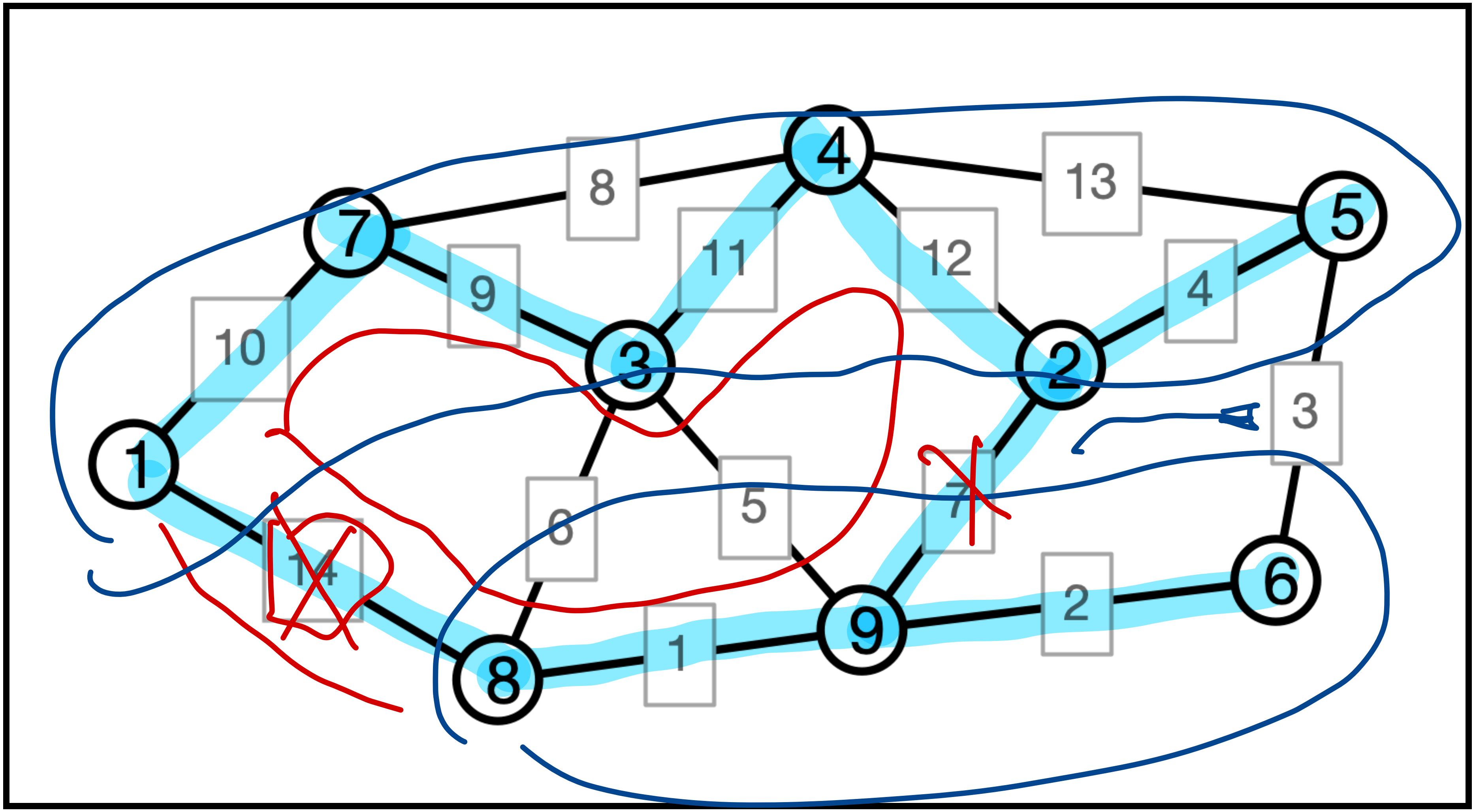
Output:

- a set F of edges in E such that

  1. (V, F) is connected
  2. sum of weights of edges in F is minimal among all connected sub-graphs of G

The graph $T = (V, F)$ is called a **minimum spanning tree** of G

Example

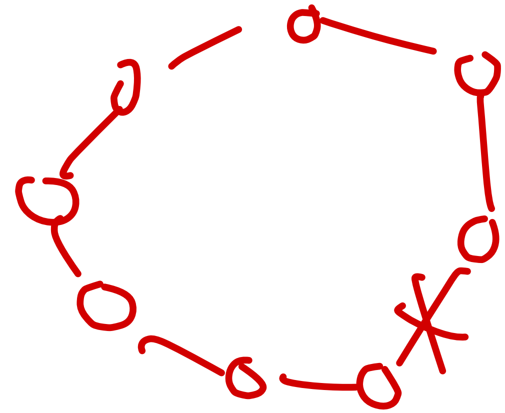


Trees

Recall. A **tree** T is a graph that:

1. is connected, and
2. contains no cycles

MST Problem



Input:

- a weighted graph $G = (V, E)$ with edge weights w

Output:

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2. sum of weights of edges in F is minimal among all connected sub-graphs of G

Question. Why must $T = (V, F)$ be a tree?

(1) T is connected

(2) If T has cycle, can remove edge from cycle w/out disconnecting T $\Rightarrow$ better sol'n

A Claim

Claim. Suppose T is an MST for a weighted graph G. Then T is a tree.

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Proof. Argue by contradiction.

- Suppose T is not a tree
- By definition of MST, T is connected
- So T must contain a cycle C

A Claim

Claim. Suppose T is an MST for a weighted graph G. Then T is a tree.

Proof. Argue by contradiction.

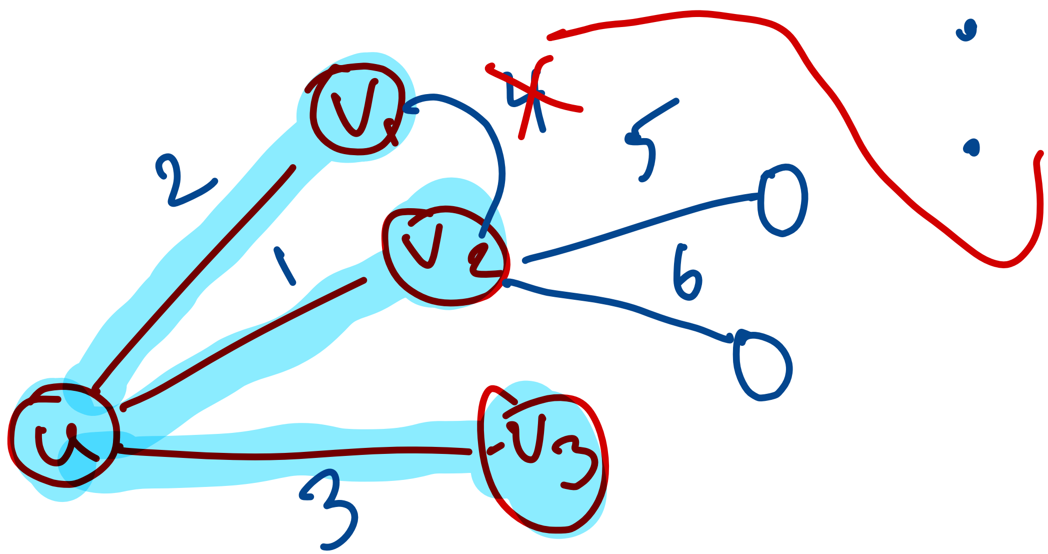
- Suppose T is not a tree
- By definition of MST, T is connected
- So T must contain a cycle C
- Removing any edge e from C results in a smaller spanning "tree"
- $\implies T$ was not minimal, a contradiction

An Idea

Grow a tree outward from a seed vertex:

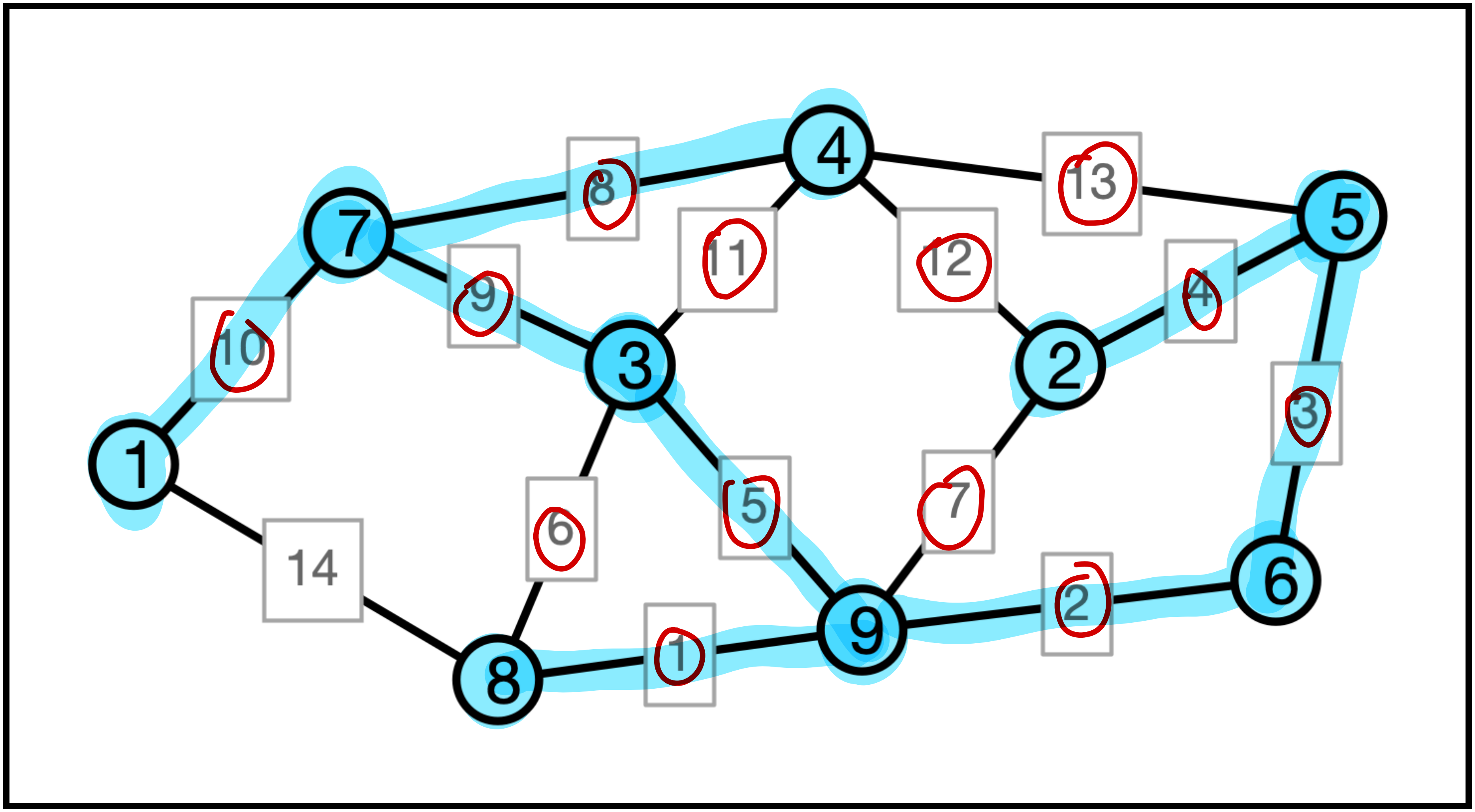
- start with u
- repeat until we get a spanning tree:
  - pick a new edge e to add to our tree

Question. How should we pick the "next" edge?



- cheap
- not redundant

Tree Growing Example



Exercise. Do we get same ans. starting from vtx other than 7 ?

Question

What information do we need to maintain in order to implement this procedure?

- set of vertices we've visited
- weights of edges

corresponding to "new" neighbors

priority queue w/ priority = weight

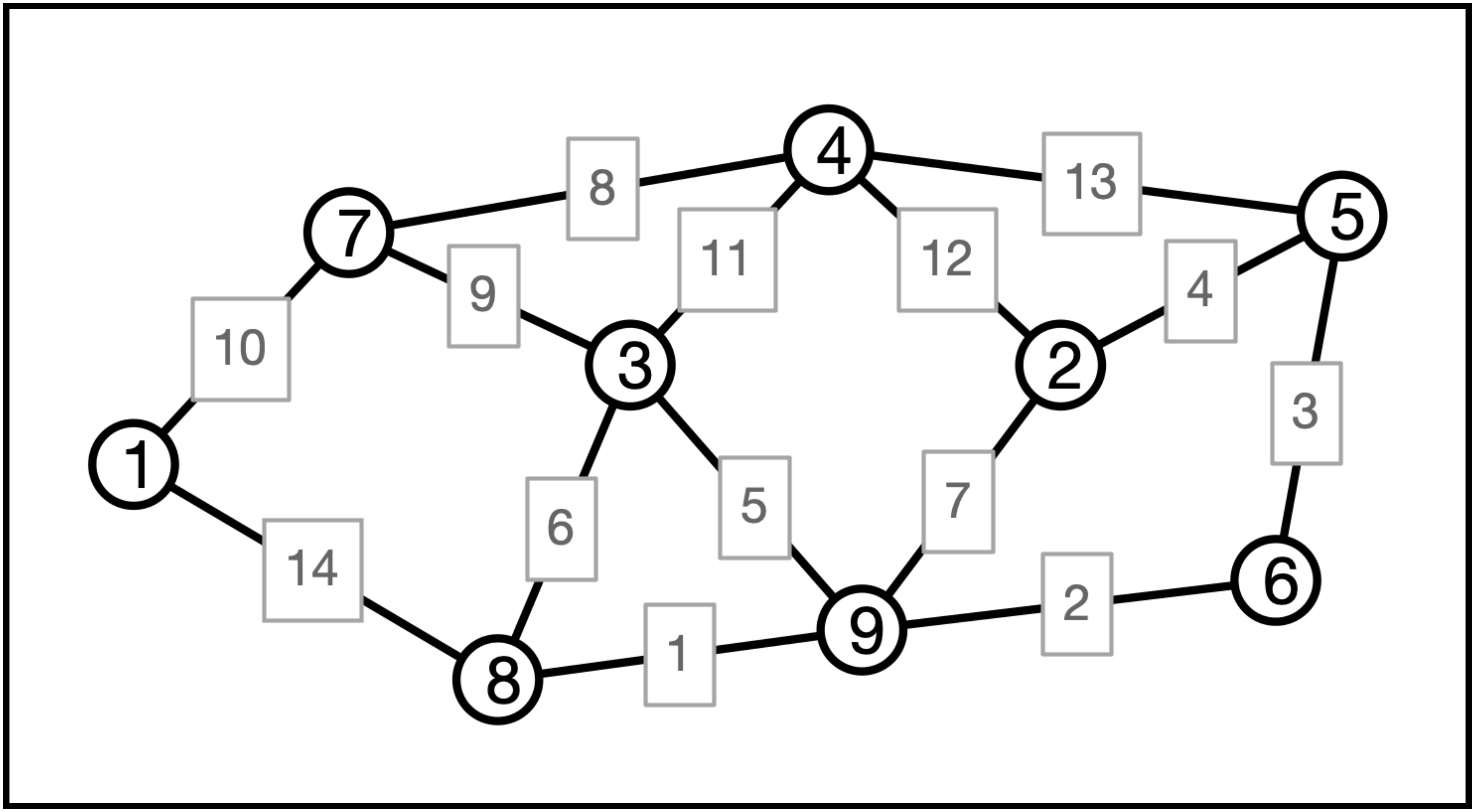
Prim's Algorithm

```
PrimMST(V, E):
  initialize set S = {v} with v arbitrary
  initialize set F = {} of MST edges, priority queue Q
  for each neighbor x of v
    add (v, x) to Q with priority w(v, x)
  while Q is not empty
    (u, v) <- removeMin(Q)
    if S doesn't contain v
      add (u, v) to F
      for each neighbor x of v
        add (v, x) to Q with priority w(v, x)
  return (S, F)
```

v is not yet connected to rest of tree

, add v to S

Prim Illustration



Question

Why is graph returned by Prim a tree?

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when add (u,v), v doesn't have neighbors in F

(u,v) doesn't create a cycle

Another Question

Why is graph returned by Prim a MST?

- assume all edge weights are distinct

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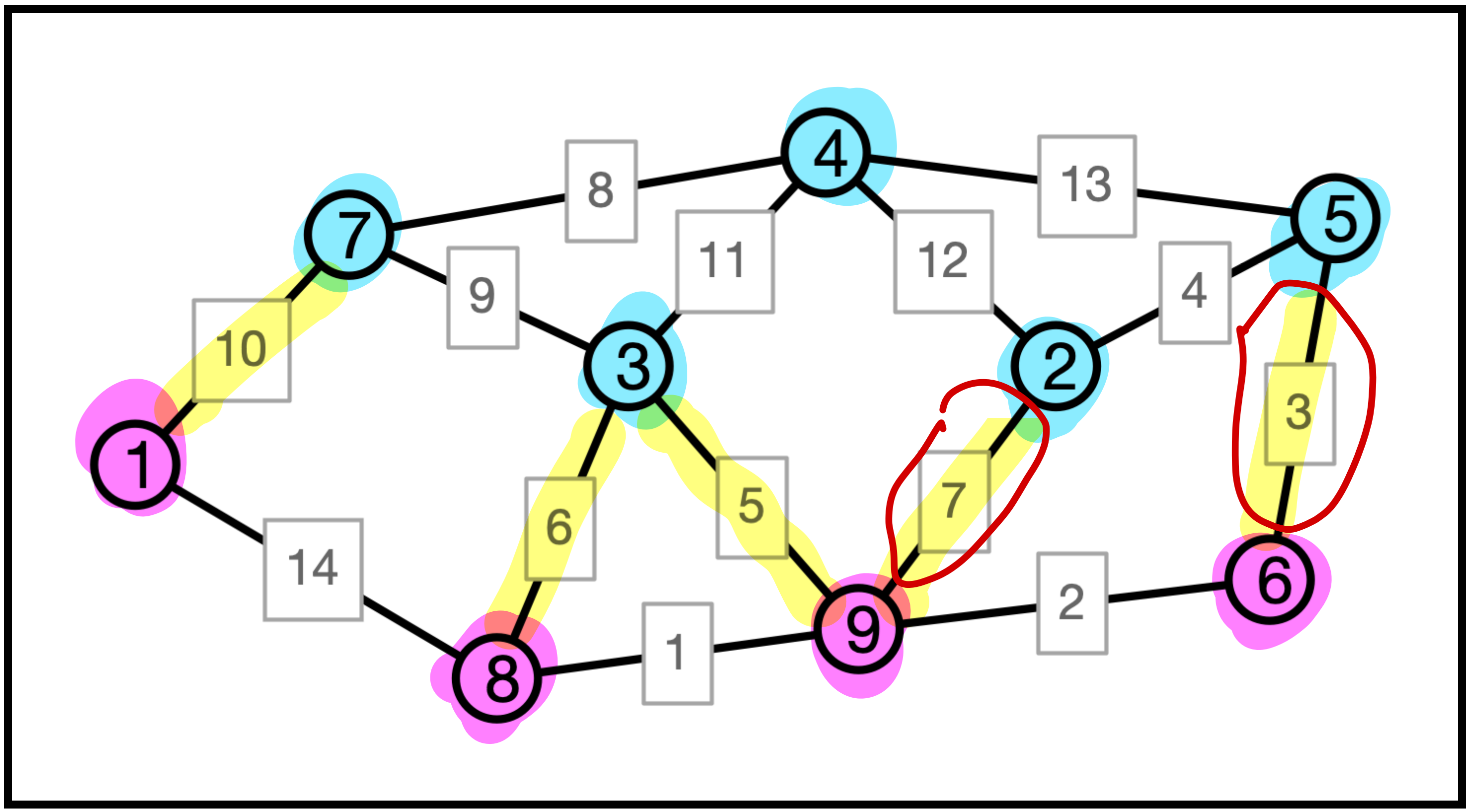
Why is graph returned by Prim a MST?

- assume all edge weights are distinct

Must show. Every edge added to F is in an MST.

Cuts in Graphs

Definition. Let $G = (V, E)$ be a graph. A **cut** in G is a partition of V into two (non-empty) subsets U and $V - U$.



cut edges are edges that "cross" the cut i.e. w/ one endpt in U other in V-U

Cuts and MSTs

Cut Claim. Suppose:

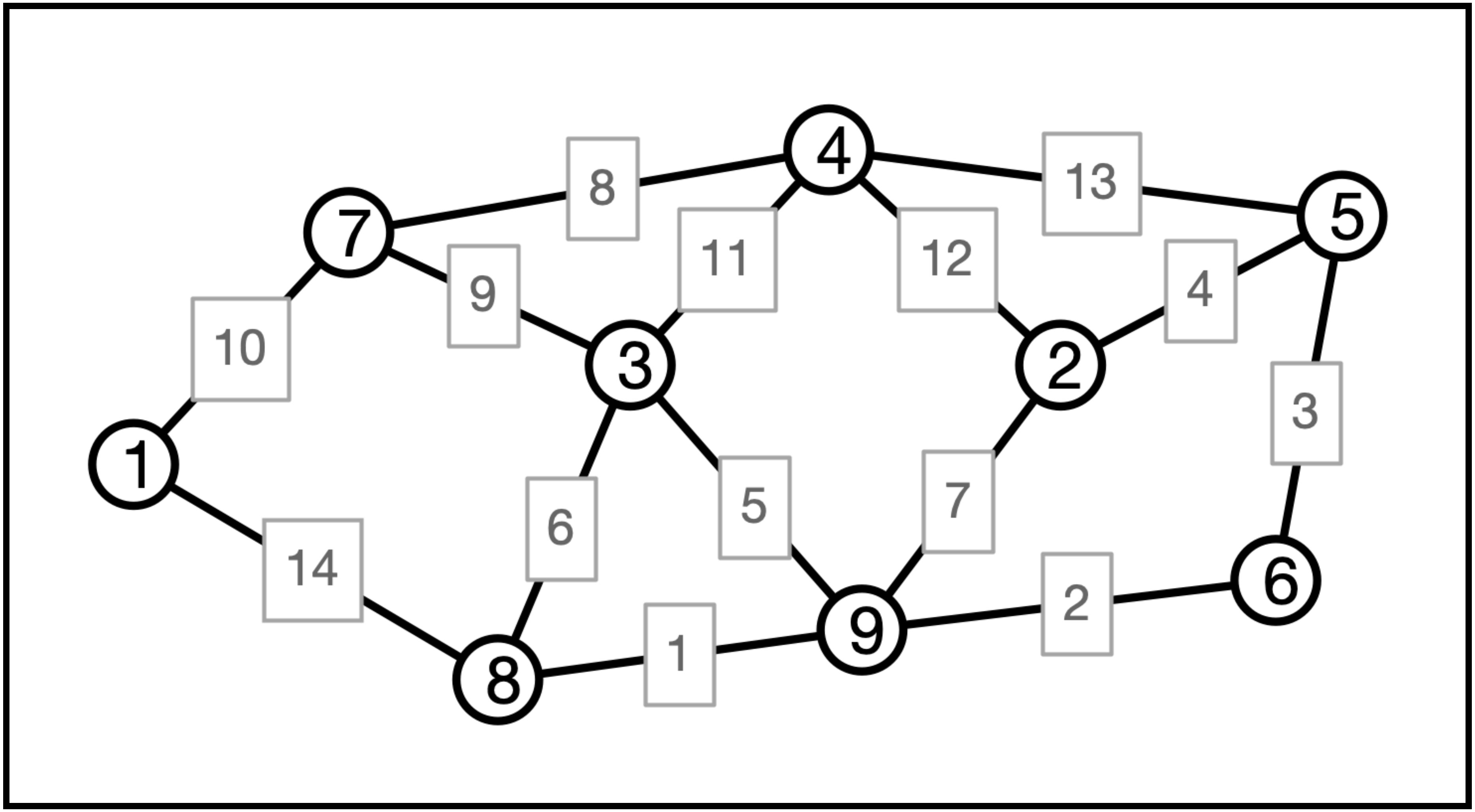
- $G = (V, E)$ is a weighted graph (with distinct edge weights)
- $U, V - U$ a cut in G
- $T = (V, F)$ an MST
- $e = (u, v)$ is the minimum weight edge that crosses the cut
  - $u \in U$ and $v \in V - U$

Then:

- T contains the edge e

Consider: Why does cut claim $\Rightarrow$ Prim give MST?

Cut Claim Illustration



Prim, Again

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Prim Correctness I

Cut claim $\implies$ Prim produces an MST.

Why?

Prim Correctness I

Cut claim $\implies$ Prim produces an MST.

Why?

Consider kth edge e_k added by Prim

- S_k = contents of S before edge added

What can we say about the cut $S_k, V - S_k$?

Prim Correctness II

First Conclusion. Every edge e added by Prim's algorithm is in the MST.

Still to show. All MST edges are added.

Why is this so?

Prim Correctness II

First Conclusion. Every edge e added by Prim's algorithm is in the MST.

Still to show. All MST edges are added.

Why is this so?

- Suffices to argue that Prim produces a spanning tree
- Set of edges found by Prim form a tree
- All vertices of V are in tree (if G is connected)

Conclusion. Prim's algorithm produces an MST.

Prim Running Time?

```
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Conclusion

Prim's algorithm:

- computes an MST in G
- if efficient priority queue is used, running time is $O(m \log n)$

Prim's algorithm is **greedy**

- to grow a tree, always add the lightest outgoing edge

Next Time

Another greedy MST algorithm!