Lecture 13: Bridges, Graphs, and Greed

COSC 311 *Algorithms*, Fall 2022

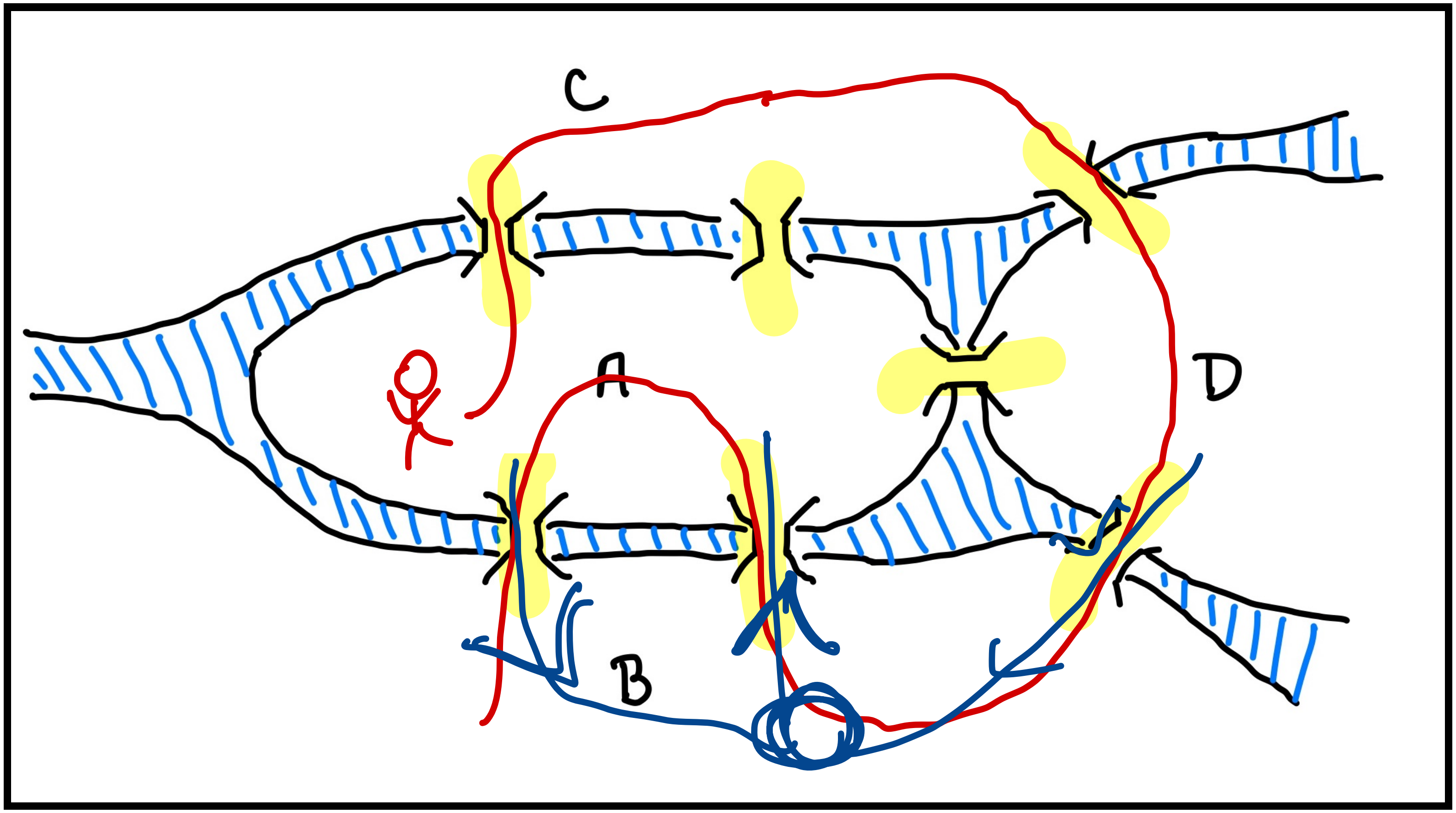
Announcements

1. Midterm next Friday
    - Material up to lecture 12 covered
    - Makeup date following week
    - Accommodations
2. Study guide posted this weekend
    - Topics
    - Example questions
    - Solutions

Overview

1. Bridges of Königsberg
2. Graphs
3. Greedy Algorithms

Bridges of Königsberg



Question. Is it possible to walk around Königsberg, cross every bridge *exactly* once, and return to where you started?

A Result

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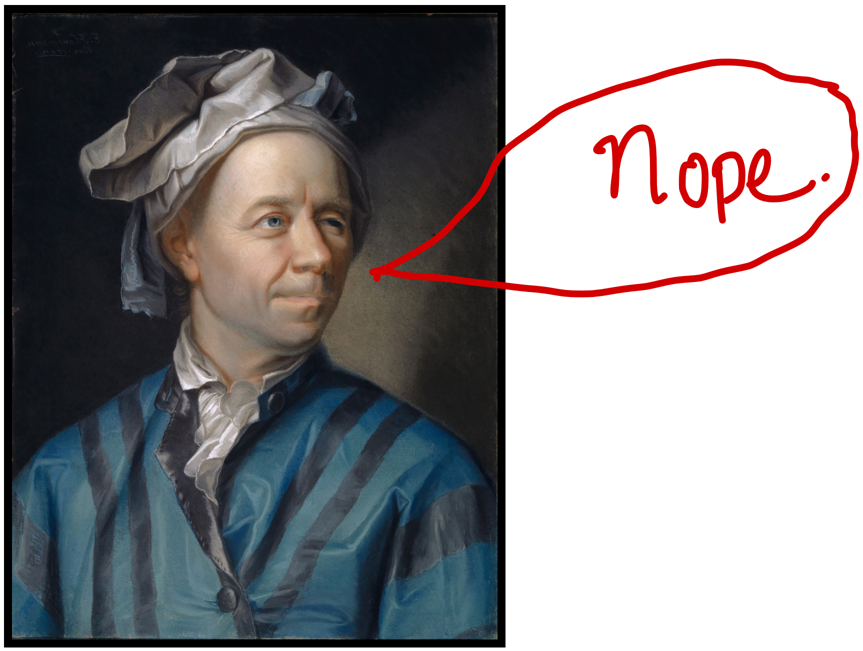
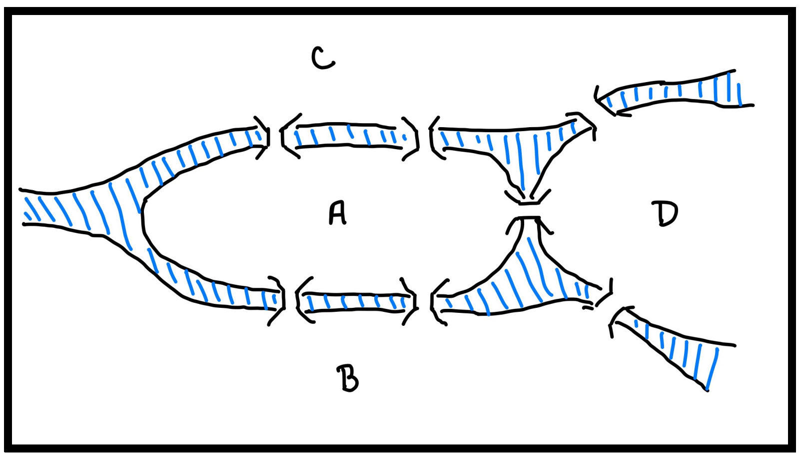
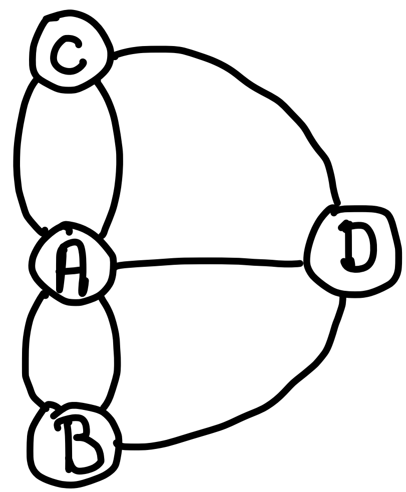


Theorem (Euler, 1736). No.

Abstraction

1. Replace each separate landmass with a single point
    - all points in each landmass are reachable from each other without crossing a bridge
2. Represent each bridge with an "edge" connecting landmasses



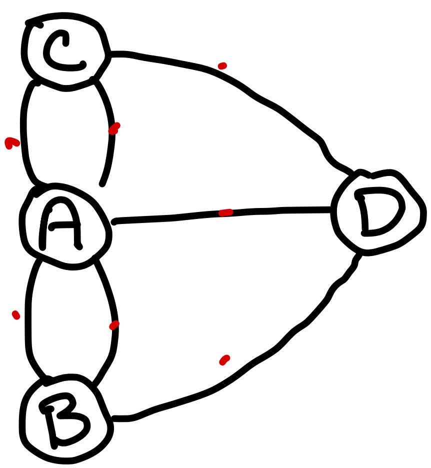


Graphs

A **graph** $G = (V, E)$ consists of

1. a set V of **vertices** or **nodes**
2. a set E of **edges**, where each edge consists of a pair of nodes
   - if $e = (u, v)$ an edge, we say u and v are **adjacent**
   - G a **multigraph** if multiple edges between same pair of vertices

Example. Königsberg graph



$V = \{A, B, C, D\}$

$E = \{(A,C), (A,C), (A,B), (A,B), (A,D), (C,D), (B,D)\}$

Paths and Circuits

Note. Terminology varies from source to source.

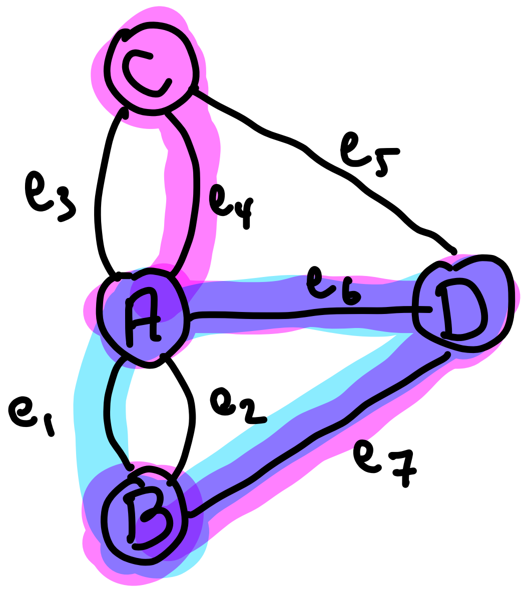
A **path** P of length k in G is a sequence of the form $v_0 e_1 v_1 e_2 v_2 \cdots e_k v_k$ where

- each v_i is a vertex, and
- each e_i is an edge with $e_i = (v_{i-1}, v_i)$.

G is **connected** if for every pair of vertices $u, v \in V$, there is a path from u to v.

P is a **circuit** if $v_0 = v_k$.

Examples



Pink Path:

$B e_7 D e_6 A e_4 C$

Length = 3

Circuit? No!

Blue Path:

$B e_1 A e_6 D e_7 B$

Length: 3

Circuit: yes.

BoK as a Graph Problem

Original Question. Is it possible to walk around Königsberg, cross every bridge *exactly* once, and return to where you started?

Rephrasing as Graph Problem. Given a graph $G = (V, E)$, is there a circuit that contains every edge $e \in E$ exactly once?

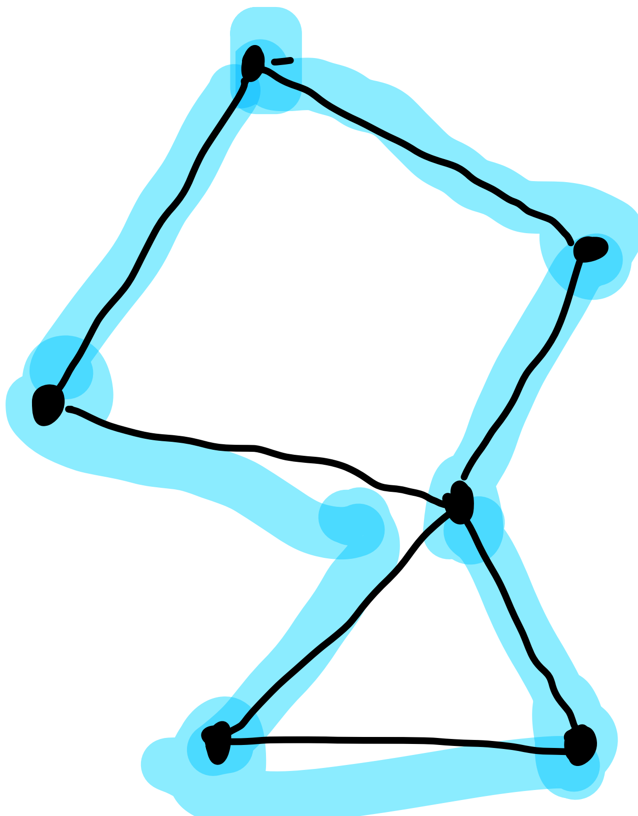
- A graph with this property is called **Eulerian**.

General Questions

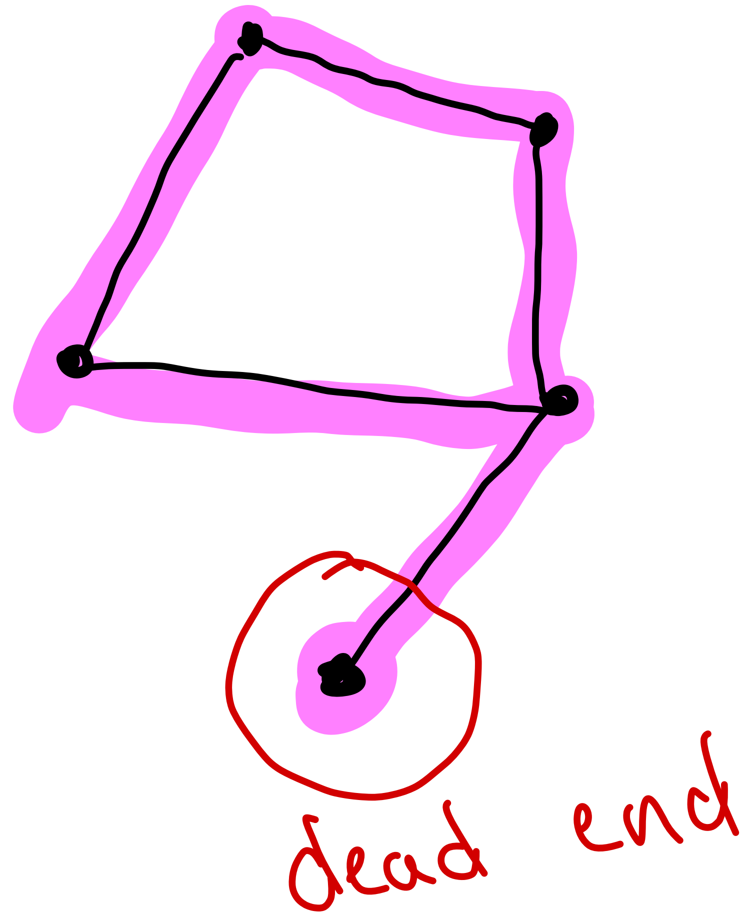
Question 1. Under what conditions is a graph G Eulerian?

Question 2. If G is Eulerian, how can we find an Eulerian circuit?

Simple Examples



Eulerian

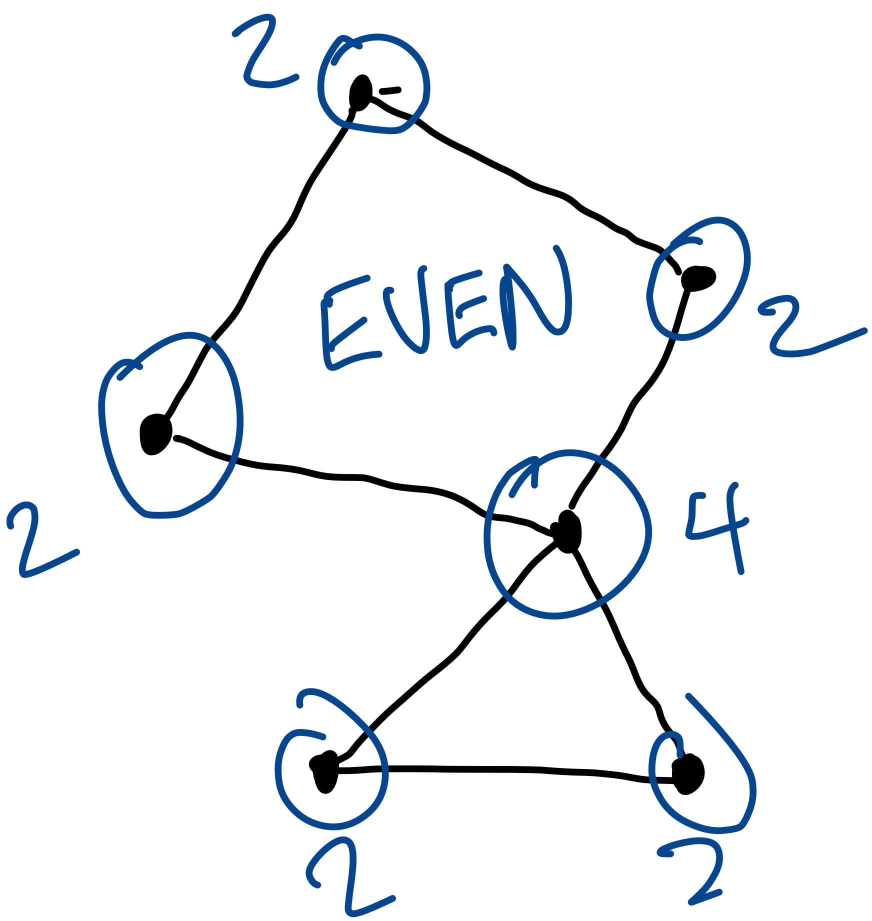


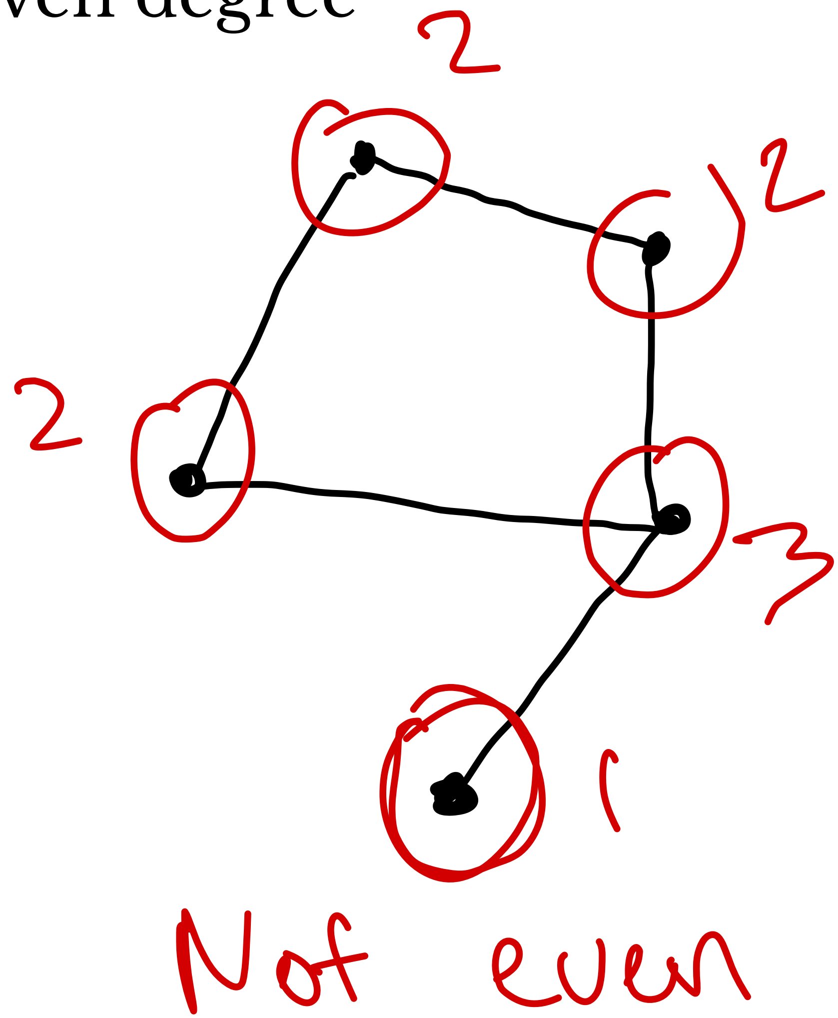
Not Eulerian

Degrees

Let $G = (V, E)$ be a graph, and $v \in V$ a vertex. The **degree** of v, $\deg(v)$ is the number of edges in E incident to v.

- G is **even** if all vertices have even degree

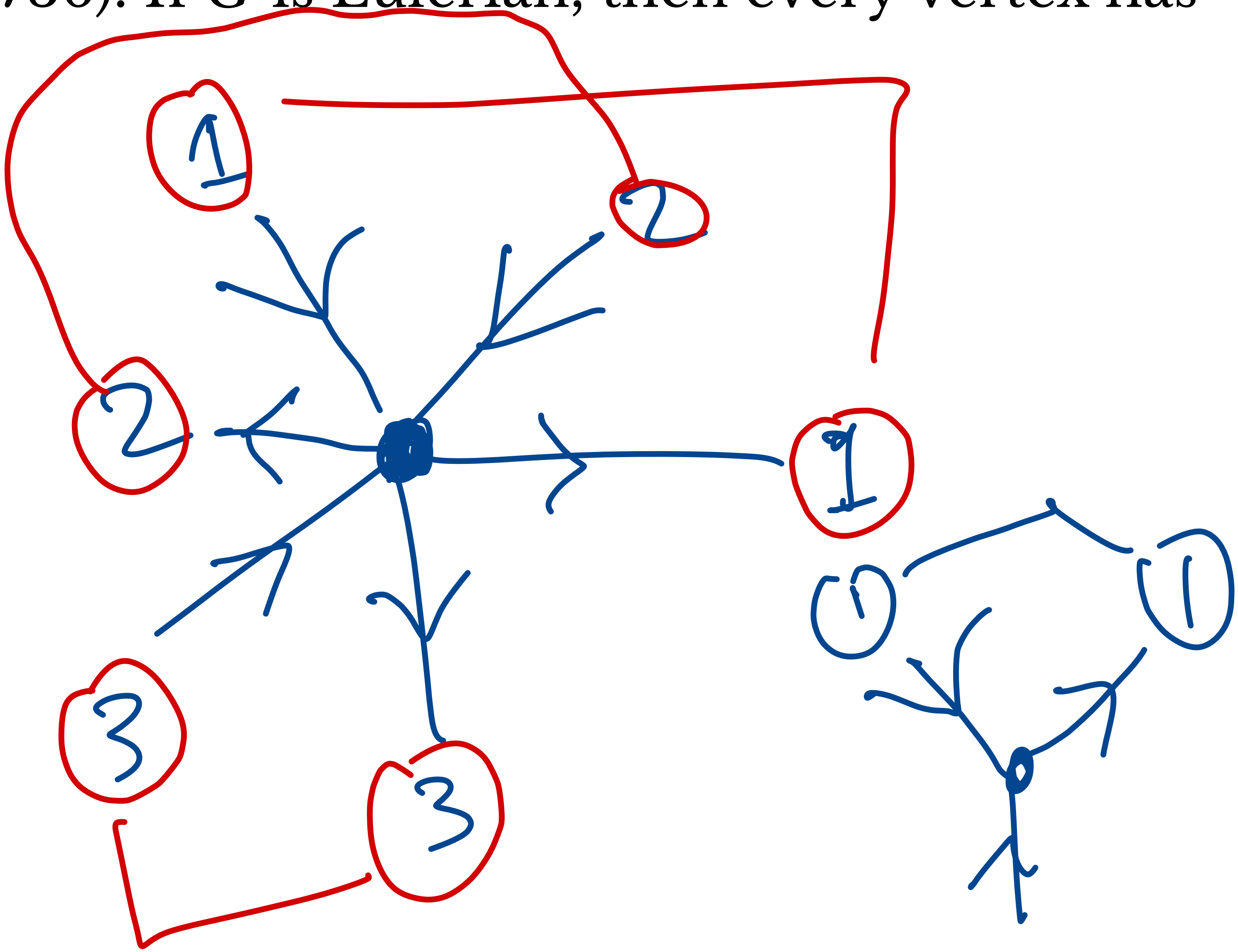




A Necessary Condition

Claim (Euler 1736). If G is Eulerian, then every vertex has *even* degree.

Why?



A Necessary Condition

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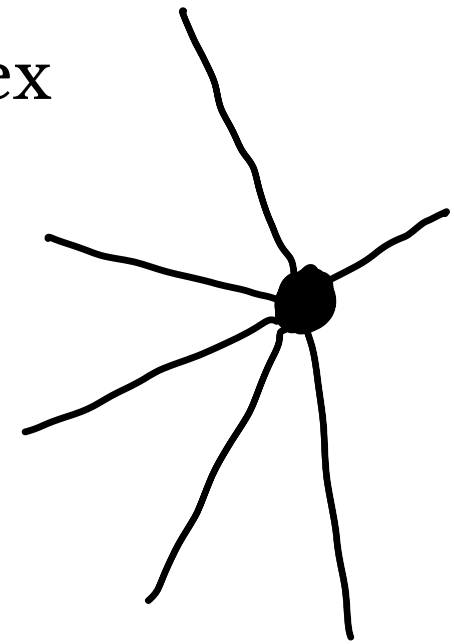
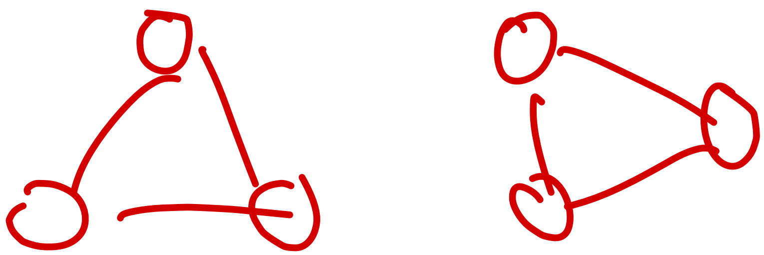
Why?

- Suppose u is a vertex other than starting vertex
- Eulerian circuit visits u total of k times
- Each visit must
  - cross one bridge to enter
  - cross another to exit
- Circuit crosses every bridge to u exactly once
- $\implies$ degree of $\deg(u) = \boxed{2k}$ even #



Question

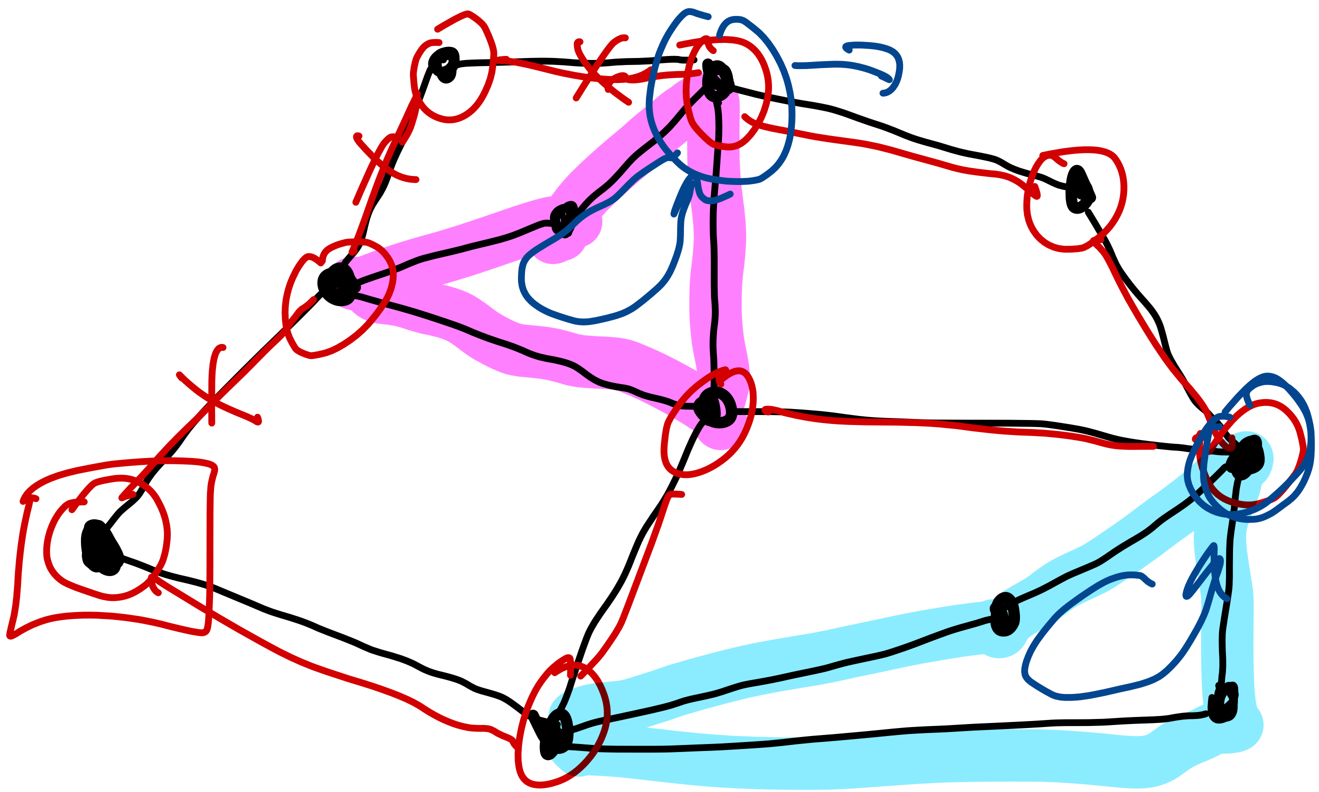
If all vertices have even degrees, is G necessarily Eulerian?



Need G to be connected,
but then maybe...

A Result

Theorem (Euler 1736). If every vertex v in a graph G has even degree and G is connected, then G is Eulerian.



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Proof technique:

- Devise an algorithm to find an Eulerian cycle

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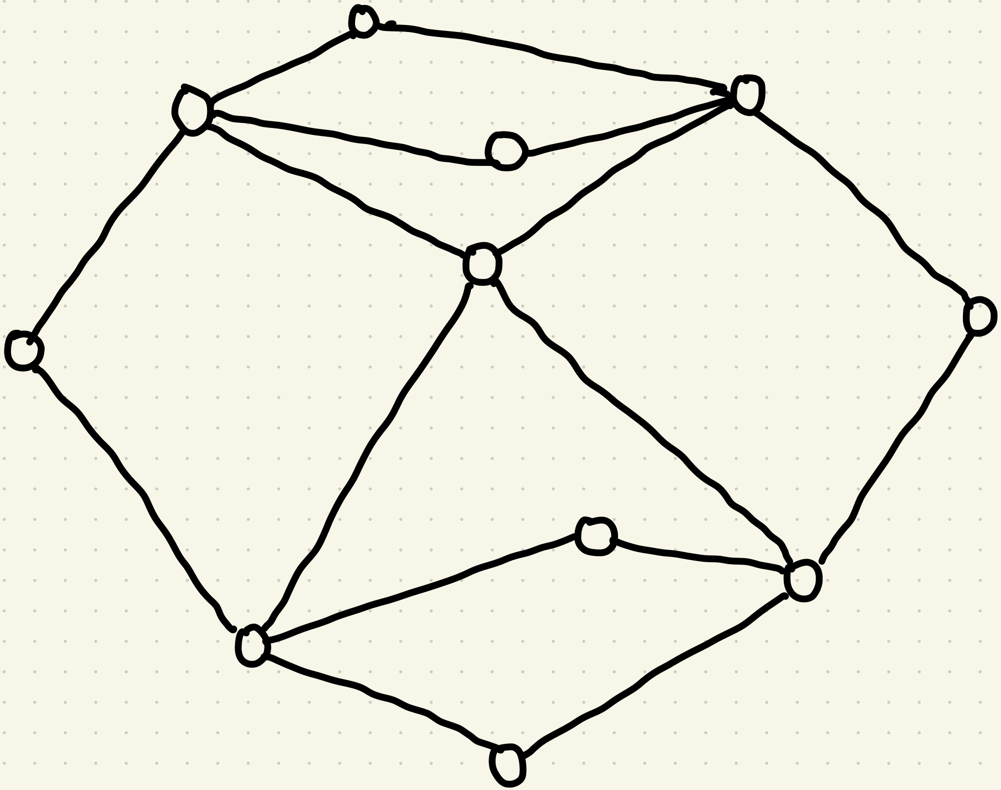
Algorithmic technique:

- Go wild!
  - wander aimlessly crossing only uncrossed bridges
    - *greedily* collect new bridges to cross
  - continue until you reach your starting point
  - reassess



Finding a Circuit

Input:

- graph G
  - set V of vertices
  - set E of edges
- starting vertex $v \in V$
- *assume* all vertices have even degree (G is **even**)

Output:

- a circuit $P = v_0 e_1 v_1 e_2 v_2 \cdots e_k v_k$ with $v_0 = v_k = v$
- every edge e incident to v is contained in P

Illustration of Technique

FindCircuit Subroutine

```
FindCircuit(V, E, v):
  cur <- v
  P <- v
  while deg(cur) > 0
    e <- any edge in E incident to cur
    (prev, cur) <- e
    append e, cur to P
    remove e from E
    if deg(prev) = 0 then remove prev from V
  endwhile
  remove cur from V
  return P
```

Circuit Finding

Claim. If every vertex in $G = (V, E)$ has even degree, then `FindCircuit(V, E, v)` returns a circuit beginning and ending at v.

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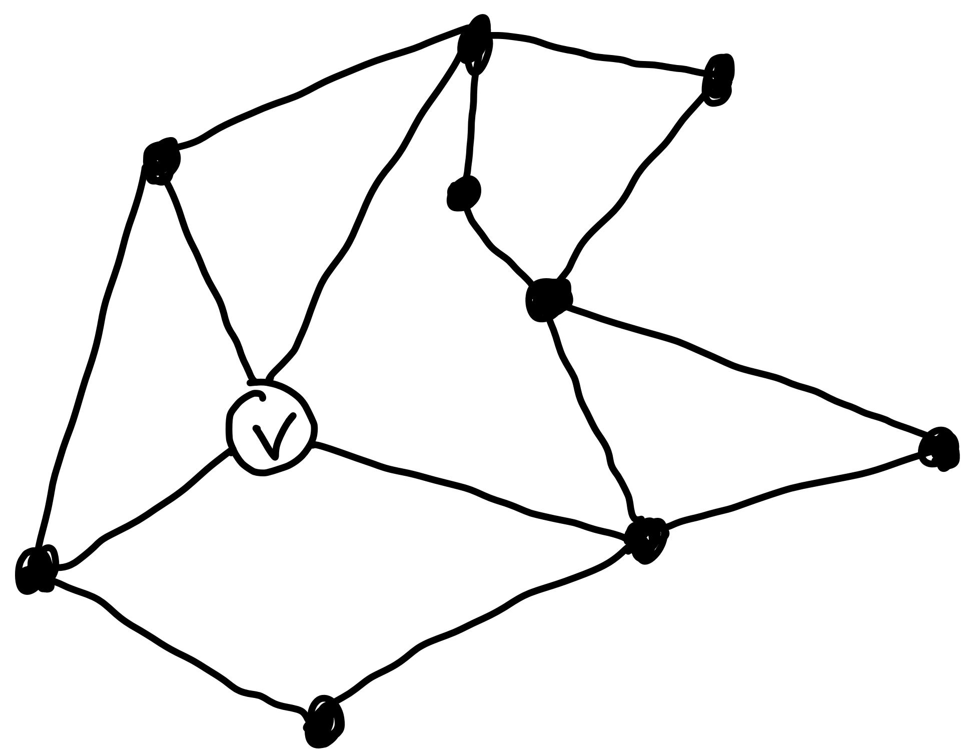
- *Loop invariant.* If cur $\neq v$, then cur and v have odd degress, while all other vertices have even degrees.



Circuit Finding

Claim. If every vertex in $G = (V, E)$ has even degree, then `FindCircuit(V, E, v)` returns a circuit beginning and ending at v.

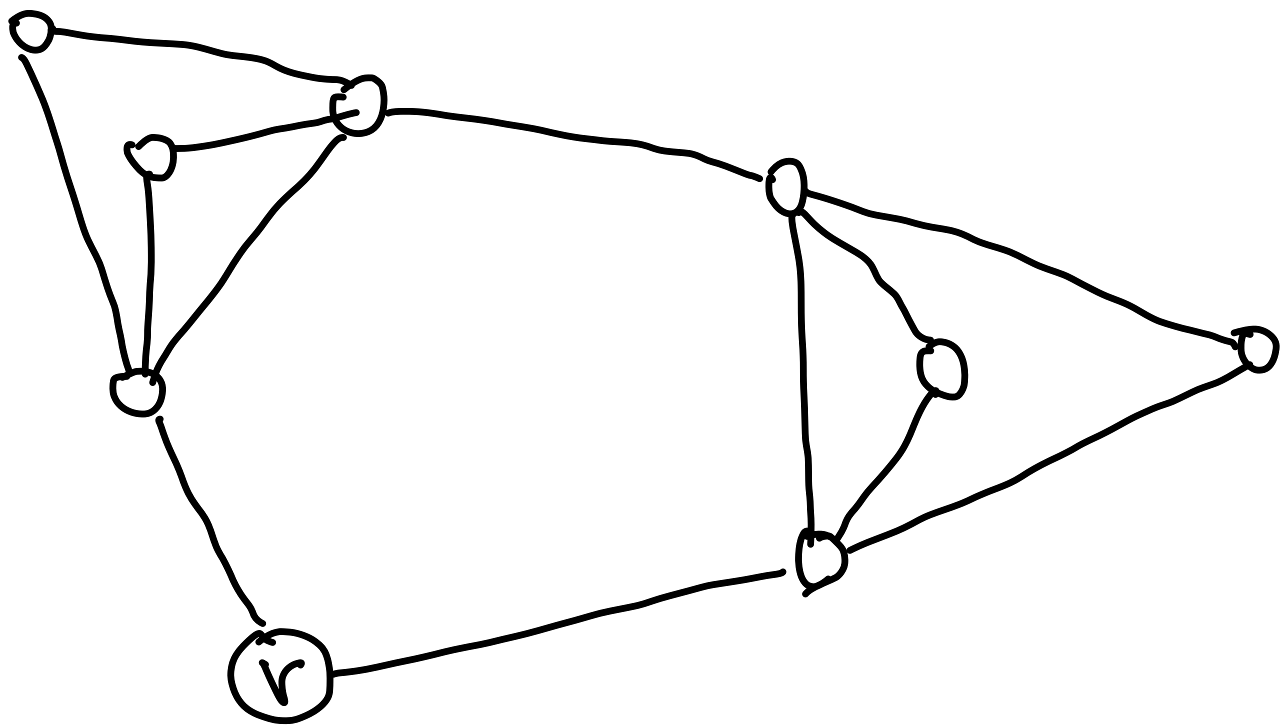
- *Loop invariant.* If cur $\neq v$, then cur and v have odd degress, while all other vertices have even degrees.
- *Consequence.* If $\deg(\text{cur}) = 0$, then cur $= v$.
  - $\implies$ can only get "stuck" at starting point!

Finding Eulerian Circuits

Strategy.

1. Apply `FindCircuit` to find a circuit $P = v_0 e_1 v_1 \cdots v_k$
2. Traverse P
    - if a vertex v_i with $\deg(v_i) > 0$ is encountered,
        1. apply `FindCircuit` to v_i to get a circuit Q
        2. splice Q into P at v_i
    - continue traversing P (with Q spliced in)

Example



Eulerian Circuit Pseudocode

```
EulerCircuit(V, E, v):
  P <- FindCircuit(V, E, v)
  for each edge e = (u, w) in P do
     if deg(w) > 0 then
       Q <- EulerianCircuit(V, E, w)
       Splice(P, Q, w)
     endif
  endfor
```

Correctness

Claim. If G is even and connected, then `EulerCircuit` returns an Eulerian circuit.

Argue by induction on $m =$ number of edges in G.

Base Case, $m = 0$. If G is connected and has no edges, then G has only one vertex, so `EulerCircuit` correctly outputs an Eulerian circuit (of length 0)

Inductive Step

Suppose `EulerCircuit` finds an Eulerian circuit on all connected, even graphs with fewer than m edges. Then:

1. After removing P from G, G has fewer than m edges
2. G is still even
3. G may be disconnected, but all components touch P
4. By inductive hypothesis, `EulerCircuit` finds Eulerian circuit in each component
5. Splicing together circuits gives Eulerian circuit for whole graph

Conclusion

G is Eulerian if and only if G is even and connected.

If G is Eulerian, then an Eulerian circuit can be found by *greedily* traversing the graph,

Next Time

More (greedy) graph algorithms!